Determining the long-distance contribution to the HLbL portion of \( g-2 \) in position space from the \( \pi^0 \) pole

Muon g-2 Theory Initiative Hadronic Light-by-Light Working Group Workshop

March 13, 2018


(RBC and UKQCD Collaborations)
Overview

• Long distance contribution to HLbL comes from the $\pi^0$ exchange.

• Calculate $\pi^0$ exchange from lattice QCD
  – Direct calculation, without form factor decomposition or parameterization.
  – Position-space based.
  – Can be applied for large volume.

• No implementation at present.
Introduce $\pi^0$ states

- Compute the $\pi^0$ pole contribution to:

$$\mathcal{A}_{\mu\mu',\nu\nu'}(x, x', y, y') = \langle 0 | T(J_\mu(x)J_{\mu'}(x')J_\nu(y)J_{\nu'}(y')) | 0 \rangle$$

- Assume $x$ and $y$ are far separated in the time direction and insert sum over $\pi^0$ states:

$$\mathcal{A}_{\mu_1\mu_2\nu_1\nu_2}^{\pi^0}(x, x', y, y') = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E_\pi(p)} \langle 0 | T(J_\mu(x)J_{\mu'}(x')) | \pi^0(p) \rangle$$

- Dominant contribution for $x_0-y_0$ large.
Use translational symmetry

\[ A_{\mu_1 \mu_2 \nu_1 \nu_2}^{\pi^0}(x, x', y, y') = \frac{1}{(2\pi)^3} \int \frac{d^3 p}{2E_\pi(p)} \langle 0 | T(J_\mu(x)J_{\mu'}(x')) | \pi^0(\vec{p}) \rangle \langle \pi^0(\vec{p}) | T(J_\nu(y)J_{\nu'}(y')) | 0 \rangle \]

\[ = \langle 0 | T(J_\mu(0)J_{\mu'}(\vec{x})) | \pi^0(\vec{p}) \rangle e^{i\vec{p} \cdot \vec{x} - E_p x_0} \]
\[ = \mathcal{F}_{\mu \mu'}(\vec{x}, \vec{p}) e^{i\vec{p} \cdot \vec{x} - E_p x_0} \]
\[ = \mathcal{F}_{\mu \mu'} \left( \vec{x}, -i \vec{\nabla}_x \right) e^{i\vec{p} \cdot \vec{x} - E_p x_0} \]

\[ E_p = \sqrt{\vec{p}^2 + M_\pi^2} \]

- After these standard steps the integral over \( \vec{p} \) can be performed, giving the Euclidean pion propagator:

\[ \frac{1}{(2\pi)^3} \int \frac{d^3 p}{2E_\pi(p)} e^{i\vec{p} \cdot (x - y) - E_p (x_0 - y_0)} = \frac{1}{(2\pi)^4} \int d^4 p \frac{e^{i p (x - y)}}{p_0^2 + \vec{p}^2 + M_\pi^2} \]
\[ \equiv \Delta_F(x - y, M_\pi) \]
Further long-distance approximation

- Combine these results to obtain:

\[ A_{\mu_{\mu}'\nu_{\nu}'}^{\pi^0}(x, x', y, y') = \mathcal{F}_{\mu_{\mu}'} \left( \tilde{x}, -i \tilde{\nabla}_x \right) \mathcal{F}_{\nu_{\nu}'} \left( \tilde{y}, i \tilde{\nabla}_y \right) \Delta_F(x - y, M_\pi) \]

- Next evaluated the spatial derivatives:

\[
\prod_{i=1}^{N} \left( \frac{\partial}{\partial x_{\rho_i}} \right) \Delta_F(x - y, M_\pi) = \left\{ \prod_{i=1}^{N} \left( \frac{\partial}{\partial x_{\rho_i}} \right) \right\} \left( \frac{M_\pi}{2\pi |x - y|} \right)^{3/2} e^{-|x-y|M_\pi} \\
\approx \left\{ \prod_{i=1}^{N} \left( -M_\pi \frac{(x - y)_{\rho_i}}{|x - y|} \right) \right\} \left( \frac{M_\pi}{2\pi |x - y|} \right)^{3/2} e^{-|x-y|M_\pi}
\]

- Which implies:

\[ A_{\mu_{\mu}'\nu_{\nu}'}^{\pi^0}(x, x', y, y') = \mathcal{F}_{\mu_{\mu}'} \left( \tilde{x}, iM_\pi \hat{n} \right) \mathcal{F}_{\nu_{\nu}'} \left( \tilde{y}, -iM_\pi \hat{n} \right) \Delta_F(x - y, M_\pi) \]

where \( \hat{n} = \frac{\tilde{x} - \tilde{y}}{|x - y|} \), a unit Euclidean four-vector.
Calculate $\gamma\gamma$ - pion vertex directly

- The amplitude $\mathcal{F}_{\mu\mu'}(\bar{x}, iM_\pi \hat{n})$ also appears in a simpler Green’s function:

\[ \mathcal{B}_{\mu\mu'}(x, x', z) = \langle 0 | T(J_\mu(x)J_{\mu'}(x')\pi^0(z)) | 0 \rangle \]

- This can be directly evaluated using lattice QCD
Calculate $\gamma \gamma$ - pion vertex directly

- The amplitude $\mathcal{F}_{\mu\mu'}(\bar{x}, iM_{\pi}\hat{n})$ also appears in a simpler Green's function:

$$\mathcal{B}_{\mu\mu'}(x, x', z) = \langle 0 | T(J_\mu(x)J_{\mu'}(x')\pi^0(z)) | 0 \rangle$$

- Involves the same $\gamma \gamma - \pi$ vertex as $A_{\mu\nu'\nu''}(x, x', y, y')$

$$\mathcal{B}_{\mu\mu'}^{\pi^0}(x, x', z) = \mathcal{F}_{\mu\mu'}(\bar{x}, iM_{\pi}\hat{n}) \ Z_{\pi^0}^{1/2} \Delta_F(x-z, M_{\pi})$$

where $Z_{\pi^0}^{1/2} = \langle \pi^0(\vec{p} = 0) | \pi^0(0) | 0 \rangle$

- Combining results for $A_{\mu_1\nu_1\nu_2}(x, x', y, y')$ and $\mathcal{B}_{\mu\mu'}^{\pi^0}(x, x', z)$

$$A_{\mu\mu'\nu\nu'}^{\pi^0}(x, x', y, y') = \mathcal{B}_{\mu\mu'}^{\pi^0}(x, x', z) \mathcal{B}_{\nu\nu'}^{\pi^0}(y, y', z') \frac{1}{Z_{\pi^0} \Delta_F(x-z, M_{\pi}) \Delta_F(z'-y, M_{\pi})} \Delta_F(x-y, M_{\pi})$$
Lattice implementation

\[ \text{Diagram showing lattice structure with points } x, y, z, \text{ and } z'. \]
\( \pi^0 \) contribution to \((g-2)_{\text{HLbL}}\)

- Combine photon and muon propagators with the QCD part to obtain:

\[
\frac{1}{2m_\mu} F_{2\pi}^\pi (q^2 = 0)(\sigma_{s',s})_i \\
= \frac{1}{VT} \sum_{x,x',y,y'} \frac{(-ie)^6}{2} \epsilon_{i,j,k} \left( x' - \frac{x + y}{2} \right)_j \\
\cdot i\bar{u}_{s'}(\bar{0}) g_{\rho,\sigma,\kappa}(x, y, y') u_s(\bar{0}) A^{\pi^0}(x, x', y, y')_{\rho,\sigma,\kappa,k}
\]

- For \(|x-y| \geq R_{\text{min}}\), replace \(A_{\mu_1\mu_2\nu_1\nu_2}(x, x', y, y')\) with \(A^{\pi^0}_{\mu_1\mu_2\nu_1\nu_2}(x, x', y, y')\) to use the \(\pi^0\) contribution at long distances.
Conclusion

• Contribution of $\pi^0$ exchange to HLbL at long distances:
  – Is well-defined in an $x$-space calculation.
  – Can be precisely computed from lattice QCD.
  – A fixed volume QCD calculation gives the $\pi^0$ HLbL contribution in increasing volume.
• Should allow the large volume systematic error to be reduced.