Disconnected quark loop contribution to Hadronic Light-by-light diagram

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Coordinate space Point photon method

- Treat all 3 photon propagators exactly (3 analytical photons), which makes the quark loop and the lepton line connected:
  - disconnected problem in Lattice QED+QCD -> connected problem with analytic photon
- QED 2-loop in coordinate space. Stochastically sample, two of quark-photon vertex location \( x, y, z \) and \( x_{op} \) is summed over space-time exactly

![Diagram showing quark-photon interactions](image)

- Short separations, \( \min[|x-z|,|y-z|,|x-y|] < R \sim O(0.5) \text{ fm} \), which has a large contribution due to confinement, are summed for all pairs
- longer separations, \( \min[|x-z|,|y-z|,|x-y|] \geq R \), are done stochastically with a probability shown above (Adaptive Monte Carlo sampling)
**HLbL point source method**  [L. Jin et al. 1510.07100]

- Anomalous magnetic moment, $F_2(q^2)$ at $q^2 \rightarrow 0$ limit

$$
\frac{F_{2cHLbL}(q^2 = 0) (\sigma_s', s)_i}{m} = \sum_{x,y,z,x_{op}} \left( \epsilon_{i,j,k} (x_{op} - x_{ref})_j \cdot i\bar{u}_{s'}(\vec{0}) \mathcal{F}_C^C(x, y, z, x_{op}) u_s(\vec{0}) \right) 
$$

- Stochastic sampling of $x$ and $y$ point pairs. Sum over $x$ and $z$.

$$
\mathcal{F}_C^C(x, y, z, x_{op}) = (-ie)^6 \mathcal{G}_{\rho, \sigma, \kappa}(x, y, z) \mathcal{H}_{\rho, \sigma, \kappa, \nu}^{C}(x, y, z, x_{op}),
$$
Conserved current & moment method

- [conserved current method at finite q^2] To tame UV divergence, one of quark-photon vertex (external current) is set to be conserved current (other three are local currents). All possible insertion are made to realize conservation of external currents config-by-config.

- [moment method , q^2→0] By exploiting the translational covariance for fixed external momentum of lepton and external EM field, q→0 limit value is directly computed via the first moment of the relative coordinate, x_{op} - (x+y)/2, one could show

\[
\frac{\partial}{\partial q_i} M_\nu(\vec{q})|_{\vec{q}=0} = i \sum_{x,y,z,x_{op}} (x_{op} - (x + y)/2)_i \times
\]

\[
x_{src} \quad y', \sigma' \quad z', \nu' \quad x', \rho' \quad x_{sink}
\]

To directly get F_2(0) without extrapolation.

Form factor : \( \Gamma_\mu(q) = \gamma_\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m_l} F_2(q^2) \)
Simulation details [RBC/UKQCD 2015]

two gauge field ensembles generated by RBC/UKQCD collaborations

Domain wall fermions: chiral symmetry at finite $a$

Iwasaki Gauge action (gluons)

- pion mass $m_\pi = 139.2(2)$ and $139.3(3)$ MeV ($m_\pi L \lesssim 4$)
- lattice spacings $a = 0.114$ and $0.086$ fm
- lattice scale $a^{-1} = 1.730$ and $2.359$ GeV (+1.0 GeV, +1.38 GeV)
- lattice size $L/a = 48$ and 64
- lattice volume $(5.476)^3$ and $(5.354)^3$ fm$^3$ (+7 fm + 9.6 fm)

**HVP**

**disconnected quark loop contribution**

- [ C. Lehner et al. (RBC/UKQCD 2015, arXiv:1512.09054, PRL) ]
- Very challenging calculation due to statistical noise
- Small contribution, vanishes in SU(3) limit, Qu+Qd+Qs = 0
- Use low mode of quark propagator, treat it exactly (all-to-all propagator with sparse random source)
- First non-zero signal

Sensitive to $m_\pi$

crucial to compute at physical mass

\[ a_{\mu}^{\text{HVP (LO) DISC}} = -9.6(3.3)_{\text{stat}}(2.3)_{\text{sys}} \times 10^{-10} \]
Current conservation & subtractions

- conservation $\Rightarrow$ transverse tensor
  \[ \Pi_{\mu\nu}(q) = (\hat{q}^2 \delta_{\mu\nu} - \hat{q}^\mu \hat{q}^\nu)\Pi(\hat{q}^2) \]
- In infinite volume, $q=0$, $\Pi_{\mu\nu}(0) = 0$
- For finite volume, $\Pi_{\mu\nu}(0)$ is exponentially small (L.Jin, use also in HLbL)
  \[ \int_V dx^4 \langle V_\mu(x)\mathcal{O}(0) \rangle = \int_V dx^4 \partial_x (x\langle V_\mu(x)\mathcal{O}(0) \rangle) \]
  \[ = \int_{\partial V} dx^3 x\langle V_\mu(x)\mathcal{O}(0) \rangle \propto L^4 \exp(-ML/2) \to 0 \]
- e.g. DWF L=2, 3, 5 fm $\Pi_{\mu\nu}(0) = 8(3)e^{-4}, 2(13)e^{-5}, -1(5)e^{-8}$
- Subtract $\Pi_{\mu\nu}(0)$ alternates FVE, and reduce stat error
  "-1" subtraction trick [Bernecker & Meyer, Maintz]:
  \[ \Pi^{\mu\nu}(q) - \Pi^{\mu\nu}(0) = \int d^4x (e^{iqx} - 1)\langle J^\mu(x)J^\nu(0) \rangle \]
Subtraction using current conservation

- From current conservation, \( \partial_\rho V_\rho(x) = 0 \), and mass gap, \( \langle x V_\rho(x) \mathcal{O}(0) \rangle \sim |x|^n \exp(-m_\pi |x|) \)

\[
\sum_x \mathcal{H}^{C}_{\rho,\sigma,\kappa,\nu}(x, y, z, x_{\text{op}}) = \sum_x \langle V_\rho(x) V_\sigma(y) V_\kappa(z) V_\nu(x_{\text{op}}) \rangle = 0
\]

\[
\sum_z \mathcal{H}^{C}_{\rho,\sigma,\kappa,\nu}(x, y, z, x_{\text{op}}) = 0
\]

at \( V \to \infty \) and \( a \to 0 \) limit (we use local currents).

- We could further change QED weight

\[
\mathcal{G}_{\rho,\sigma,\kappa}^{(2)}(x, y, z) = \mathcal{G}_{\rho,\sigma,\kappa}^{(1)}(x, y, z) - \mathcal{G}_{\rho,\sigma,\kappa}^{(1)}(y, y, z) - \mathcal{G}_{\rho,\sigma,\kappa}^{(1)}(x, y, y) + \mathcal{G}_{\rho,\sigma,\kappa}^{(1)}(y, y, y)
\]

without changing sum \( \sum_{x,y,z} \mathcal{G}_{\rho,\sigma,\kappa}(x, y, z) \mathcal{H}^{C}_{\rho,\sigma,\kappa,\nu}(x, y, z, x_{\text{op}}) \).

- Subtraction changes discretization error and finite volume error.

- Similar subtraction is used for HVP case in TMR kernel, which makes FV error smaller.

- Also now \( \mathcal{G}_{\sigma,\kappa,\rho}^{(2)}(z, z, x) = \mathcal{G}_{\sigma,\kappa,\rho}^{(2)}(y, z, z) = 0 \), so short distance \( \mathcal{O}(a^2) \) is suppressed.

- The 4 dimensional integral is calculated numerically with the CUBA library cubature rules. \((x, y, z)\) is represented by 5 parameters, compute on \( N^5 \) grid points and interpolates. \((|x - y| < 11 \text{ fm})\).
Dramatic Improvement!

Luchang Jin

\[ a = 0.11 \text{ fm}, \ 24^3 \times 64 \ (2.7 \text{ fm})^3, \]
\[ m_\pi = 329 \text{ MeV}, \ m_\mu = \sim 190 \text{ MeV}, \ e = 1 \]

\[ q = 2\pi/L \ N_{\text{prop}} = 81000 \]
\[ q = 0 \ N_{\text{prop}} = 26568 \]

\[ 0.0825(32) \]
\[ 0.0804(15) \]

\[ \begin{array}{|c|c|c|c|c|}
\hline
\text{Method} & \frac{F_2}{(\alpha/\pi)^3} & N_{\text{conf}} & N_{\text{prop}} & \sqrt{\text{Var}} \\
\hline
\text{Conserved} & 0.0825(32) & 12 & (118 + 128) \times 2 \times 7 & 0.65 \\
\text{Mom.} & 0.0804(15) & 18 & (118 + 128) \times 2 \times 3 & 0.24 \\
\hline
\end{array} \]
SU(3) hierarchies for d-HLbL

- At $m_s = m_{ud}$ limit, following type of disconnected HLbL diagrams:
  \[ Q_u + Q_d + Q_s = 0 \]  [Mainz, Lehenr for HVP]
- Other diagrams suppressed by:
  \[ O(m_s - m_{ud})/3, O((m_s - m_{ud})^2), \text{ and } O((m_s - m_{ud})^3) \]

\[(m_s - m_{ud})^0\]

\[(m_s - m_{ud})^2\]

\[(m_s - m_{ud})^3\]
Disconnected calculation

- We can use two point source photons at $y$ and $z$, which are chosen randomly. The points $x_{\text{op}}$ and $x$ are summed over exactly on lattice.

- Only point source quark propagators are needed. We compute $M$ point source propagators and all $M^2$ combinations of them are used to perform the stochastic sum over $r = z - y$.

\[
\mathcal{F}_\nu^D(x, y, z, x_{\text{op}}) = (-ie)^6 G_{\rho,\sigma,\kappa}(x, y, z) \mathcal{H}_{\rho,\sigma,\kappa,\nu}^D(x, y, z, x_{\text{op}}) \tag{13}
\]

\[
\mathcal{H}_{\rho,\sigma,\kappa,\nu}^D(x, y, z, x_{\text{op}}) = \left\langle \frac{1}{2} \Pi_{\nu,\kappa}(x_{\text{op}}, z) [\Pi_{\rho,\sigma}(x, y) - \Pi_{\rho,\sigma}^{\text{avg}}(x - y)] \right\rangle_{\text{QCD}} \tag{14}
\]

\[
\Pi_{\rho,\sigma}(x, y) = -\sum_q \left( \frac{e_q}{e} \right)^2 \text{Tr}[\gamma_{\rho} S_q(x, y) \gamma_{\sigma} S_q(y, x)]. \tag{15}
\]
Disconnected calculation

\[
\frac{F_2^{dHLbL}(0)}{m} \left( \sigma_{s',s} \right)_i = \sum_{r,x}^\infty \sum_{x_{op}}^\infty \frac{1}{2} \epsilon_{i,j,k}(x_{op})_j i \bar{u}_{s'}(0) F^D_k(x, y = r, z = 0, x_{op}) u_s(0) \quad (16)
\]

\[
H^D_{\rho,\sigma,\kappa,\nu}(x, y, z, x_{op}) = \left\langle \frac{1}{2} \Pi_{\nu,\kappa}(x_{op}, z) \left[ \Pi_{\rho,\sigma}(x, y) - \Pi_{\rho,\sigma}^{\text{avg}}(x - y) \right] \right\rangle_{\text{QCD}} \quad (17)
\]

\[
\sum_{x_{op}}^\infty \frac{1}{2} \epsilon_{i,j,k}(x_{op})_j \left\langle \Pi_{\rho,\sigma}(x_{op}, 0) \right\rangle_{\text{QCD}} = \sum_{x_{op}}^\infty \frac{1}{2} \epsilon_{i,j,k}(-x_{op})_j \left\langle \Pi_{\rho,\sigma}(-x_{op}, 0) \right\rangle_{\text{QCD}} = 0
\]

- Because of the parity symmetry, the expectation value for the left loop average to zero.
- \([\Pi_{\rho,\sigma}(x, y) - \Pi_{\rho,\sigma}^{\text{avg}}(x - y)]\) is only a noise reduction technique. \(\Pi_{\rho,\sigma}^{\text{avg}}(x - y)\) should remain constant throughout out the entire calculation.
\textbf{M}^2 \textbf{trick}

- For \textbf{QED}_L, we can compute the QED function for all $z$ given the $y$ location fixed and $x$ summed over. Allow us to compute all combination of $y,z$ with little efforts.

- For \textbf{QED}_\infty, although we can compute all the function $G_{\rho,\sigma,\kappa}(x, y, z)$ simply by interpolate, we cannot easily compute this function (even after fixing $y$) for all $x$ and $z$, simply because of its cost is proportion to $Volume^2$.

- However, we with \textbf{QED}_\infty and interpolation, we can freely choose which coordinates we compute. For example, we may compute all $z$ for $|x - y| \leq 5$, and sample $z$ for $|x - y| > 5$. 
140 MeV Pion, disconnected (and connected) LbL results


- left: Integrand function ,

![Integrand function graph](image)

- right: Integral

![Integral graph](image)

- Using AMA with 2,000 zMobius low modes, AMA

( statistical error only )

\[
\begin{align*}
\frac{g_\mu - 2}{2} & \bigg|_{cHLbL} = (0.0926 \pm 0.0077) \times \left( \frac{\alpha}{\pi} \right)^3 = (11.60 \pm 0.96) \times 10^{-10} \\
\frac{g_\mu - 2}{2} & \bigg|_{dHLbL} = (-0.0498 \pm 0.0064) \times \left( \frac{\alpha}{\pi} \right)^3 = (-6.25 \pm 0.80) \times 10^{-10} \\
\frac{g_\mu - 2}{2} & \bigg|_{HLbL} = (0.0427 \pm 0.0108) \times \left( \frac{\alpha}{\pi} \right)^3 = (5.35 \pm 1.35) \times 10^{-10}
\end{align*}
\]
**cHLbL Different lattice spacings**

**cHLbL: lattice spacing effect** (preliminary)

1/a = 2.37 GeV, 1.73 GeV, 1.0 GeV

- Add new $24^3$, 1 GeV, ID ensemble (green)
- I and ID slightly different, but disc. errors similar
- Collecting more statistics (9 configs)

- Significant increase as $a \to 0$
dHLbL Different lattice spacings

dHLbL contribution: lattice spacing effect (preliminary)

- Large negative increase tends to cancel connected one
- Collecting more statistics!

1/a = 2.37 GeV, 1.73 GeV, 1.0 GeV
These are the subleading disconnected diagrams in the SU(3) limit.

The right diagram has a factor of $1/3$ suppression from the multiplicity of the diagram compare with the left diagram, i.e. the external photon is more likely to be on the loop with three photons.

For the left diagram, the moment method works just like the connected case. With both QED$_L$ or QED$_\infty$, we can sample $x, y$ and sum over $z$. We can use the $M^2$ trick for the $x, y$ sampling. Low-modes-averaging for the loop with $z$.

For the right diagram, The moment method still works, however, we have to use a point on the other loop as the reference point, which may be more noisy. But as mentioned above, the right diagram is more suppressed.
Summary

- Lattice calculation for g-2 calculation is improved very rapidly

- **HLbL**  [Luchang Jin et al]
  - computing leading disconnected diagrams:
    -> 8% stat error in connected, 13% stat error in leading disconnected
  - coordinate-space integral using analytic photon propagator with adaptive probability (point photon method), config-by-config conserved external current
  - take moment of relative coordinate to directly take $q \rightarrow 0$
  - AMA, zMobius, 2000 low modes
  - Infinite volume / continuum QED weight function to avoid power-like FV

- **Goal**: HLbL 10% error

  Can we see the next physics Revolution (c.f GW)?
Thank you!
HLbL from Models

- Model estimate with non-perturbative constraints at the chiral / low energy limits using anomaly: $(9-12) \times 10^{-10}$ with 25-40% uncertainty

\[
a^\text{exp}_\mu - a^\text{SM}_\mu = 28.8(6.3)_\text{exp}(4.9)_\text{SM} \times 10^{-10} \quad [3.6\sigma]
\]

F. Jegerlehner, $\times 10^{11}$

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<th>Contribution</th>
<th>BPP</th>
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<th>KN</th>
<th>MV</th>
<th>PdRV</th>
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Hadronic Light-by-Light (HLbL) contributions

Model calculations: $(105^{+26}_{-26}) \times 10^{11}$

[Prades et al., 2009, Benayoun et al., 2014]

Model systematic errors difficult to quantify

Dispersive approach difficult, but progress is being made

[Colangelo et al., 2014b, Colangelo et al., 2014a, Pauk and Vanderhaeghen, 2014b, Pauk and Vanderhaeghen, 2014a, Colangelo et al., 2015]

First non-PT QED+QCD calculation

[Blum et al., 2015]

Very rapid progress with Pert. QED+QCD

[Jin et al., 2015]