Collectivity in pp, pA, AA
Lecture 2 in Storrs

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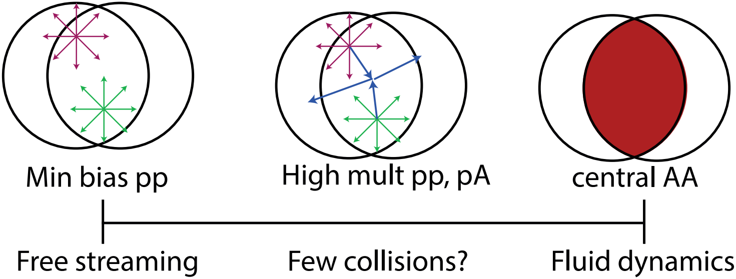
Dynamical models of collectivity



- Is that the dynamical picture of collectivity?
- Let's start with the simplest case and discuss in Lecture 3 whether/where it applies

How do we probe the QGP?

- **Inject** perturbations $\delta h^{\mu\nu}$ (energy) or δA^{μ} (charge)
  - in form of spatial eccentricities
  - as jets
  - as electric charge, baryon number, heavy flavor
- **Measure** their propagation

$$\delta T^{\mu\nu}(t,x) = \int_{t_i,x_i} G_R^{\mu\nu,\alpha\beta}(t-t_i, x-x_i)\delta h_{\alpha\beta}(t_i,x_i)$$

- **Conclude** about matter properties. How ?

$$G_R^{\alpha\beta,\gamma\delta}(t,k) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} G_R^{\alpha\beta,\gamma\delta}(\omega,k)$$

$G_R(\omega,k)$ informs us about QGP constituents and how they propagate.

A primer about kinetic theory

- Start with a system that is in global thermal equilibrium
- Push the system out of equilibrium by an external source
  - Couple to external EM-field: A^{μ}
  - Couple with gravity: $g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}(t)$

A primer about kinetic theory

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- Follow time evolution of the conserved currents in the system

$$T^{\mu\nu}(t) \quad J^\mu(t)$$

  - Expect that at late times $t \gg t_{\mathrm{micro}}$, evolution approaches hydrodynamics (we want to derive this)
  - At earlier times something else, non-hydrodynamical evolution

For a small perturbation, linear response is given by the retarded correlation function

$$J^{\mu}(x,t) = \int d^3x' dt G_J^{\mu\nu}(\mathbf{x},t;\mathbf{x}',t')\delta A_{\nu}(\mathbf{x}',t')\,,$$
$$T^{\mu\nu}(x,t) = \int d^3x' dt G_T^{\mu\nu,\alpha\beta}(\mathbf{x},t;\mathbf{x}',t')\delta h_{\alpha\beta}(\mathbf{x}',t')\,,$$

The retarded response functions carry all information on how small perturbations propagate.

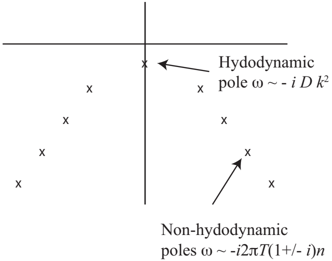
Response functions in Fourier space

$$J^{\mu}(\omega, k) = G_J^{\mu\nu}(\omega, k) A_{\nu}(\omega, k)$$
$$T^{\mu\nu}(\omega, k) = G_T^{\mu\nu,\alpha\beta}(\omega, k) h_{\alpha\beta}(\omega, k)$$

- No mode-mixing on the linear level
- Corresponds to a plane wave perturbation $k||z$
  - A^{μ} electric charge diffusion mode
  - $h_{00}, h_{xx}, h_{yy}, h_{zz}, h_{0z}$, sound mode
  - $h_{0x}, h_{0y}, h_{zy}, h_{zx}$, shear mode
  - h_{xy}, tensor mode

Non-analytic structures in $G(\omega, k)$

Example AdS/CFT: Son, Starinets JHEP 0209 (2002), Starinets PRD 66 (2002)



$$G_T^{0x,0x}(t,k) = \mathrm{Res}(\omega_{\mathrm{hydro}}) e^{-i\omega_{\mathrm{hydro}} t} + \sum_n \mathrm{Res}(\omega_n) e^{-i\omega_n t}$$

- Note: the structures must always come in pairs for the response to be real in time-domain

What happens in free kinetic theory?

$$\underbrace{p^\mu \partial_\mu f(\mathbf{x}, \mathbf{p}, t)}_{\text{free streaming}} + \underbrace{F^i \nabla_i^{(p)} f(\mathbf{x}, \mathbf{p}, t)}_{\text{external force}} = 0$$

- On-shell particles: $p^0 = |\mathbf{p}|$ ($m = 0$ for simplicity)

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- Force:

$$F^i = gF^{i\beta}p_\beta = g(\mathbf{E}\cdot\mathbf{v} + \mathbf{B}\times\mathbf{v}) \quad \text{for electro-mag}$$
$$F^i = -G\Gamma^i_{\beta\gamma}p^\beta p^\gamma \quad \text{for gravity}$$

with

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta})$$
$$\approx \frac{1}{2}\eta^{\mu\nu}(\partial_\alpha h_{\nu\beta} + \partial_\beta h_{\nu\alpha} - \partial_\nu h_{\alpha\beta})$$

What happens in free kinetic theory? - cont'd

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Fourier transform:

$$f(\omega, \mathbf{k}, p) = \int \frac{d\omega}{2\pi} \frac{d^3k}{2\pi} e^{-i\omega t + i\mathbf{k}\cdot x} f(t, \mathbf{x})$$

$$(-i\omega + i\mathbf{k}\cdot\mathbf{v})f + \frac{1}{p} F^i \nabla_i^{(p)} f = 0$$

With $\mathbf{v} = \mathbf{p}/p$.

What happens in free kinetic theory? - cont'd

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With $\mathbf{v} = \mathbf{p}/p$. Linearize in force around thermal equilibrium:

$F^i = \delta F^i,\ f = f_{\mathrm{eq}} + \delta f$

$$\delta f = -g^2 i \frac{\frac{1}{p} F^i \nabla_i^{(p)} f}{\omega - \mathbf{k}\cdot\mathbf{v} + i\epsilon}$$

$i\epsilon$ to choose the retarded, not advanced solutions

Correlation function for currents: (here electric current for simplicity)

$$\delta J^\mu = g \int_p p^\mu \delta f = -ig \int_p \frac{\frac{p^\mu}{p} F^i \nabla_i^{(p)} f}{\omega - \mathbf{k} \cdot \mathbf{v} + i\epsilon}$$

Use that in isotropic system $\frac{\partial}{\partial p^\alpha} f(p) = \frac{\partial p}{\partial p^\alpha} f'(p) = \frac{p^\alpha}{p} f'(p)$

$$\delta J^\mu = -g^2 \frac{4\pi}{(2\pi)^3} \int dp p^2 f'(p) \times \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \frac{v_\alpha F^\alpha}{\omega - \mathbf{k} \cdot \mathbf{v} + i\epsilon}$$

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$$\delta J^\mu = \underbrace{g^2 \frac{8\pi}{(2\pi)^3} \int dppf(p)}_{\text{Screening mass } m^2} \times i \underbrace{\int \frac{d\Omega_{\mathbf{v}}}{4\pi} \frac{v_\alpha F^\alpha}{\omega - \mathbf{k}\cdot\mathbf{v} + i\epsilon}}_{\text{angular structure}}$$

This is the famous HTL polarization tensor

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This is the famous HTL polarization tensor

$$\delta T^{\mu\nu} = 3G^2(\epsilon + P) i \int \frac{d\Omega}{4\pi} \frac{-\Gamma^0_{\alpha\beta} v^\alpha v^\beta}{\omega - \mathbf{k}\cdot\mathbf{v} + i\epsilon}$$

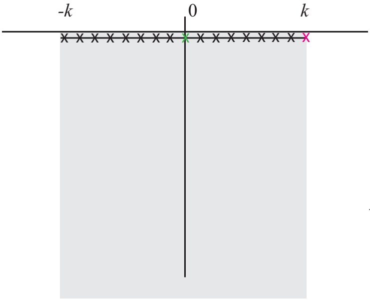
Ballistic propagator and branch cut

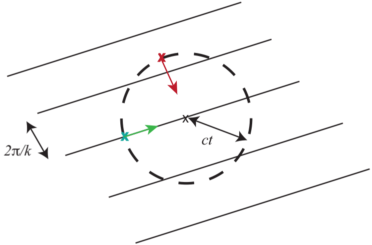
$\int \frac{d\Omega_{\mathbf{v}}}{4\pi} \frac{v_\alpha F^\alpha}{\omega - \mathbf{k}\cdot\mathbf{v} + i\epsilon}$ contains the ballistic propagator

$$\int \frac{d\omega}{2\pi} \frac{d^3k}{(2\pi)^3} \frac{e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}}{\omega - \mathbf{k}\cdot\mathbf{v} + i\epsilon} = \int \frac{d^3k}{(2\pi)^3} \theta(t) e^{i\mathbf{k}\cdot\mathbf{v}t - i\mathbf{k}\cdot\mathbf{x}} = \delta(\mathbf{v}t - \mathbf{x})\theta(t)$$

$$\begin{aligned} i \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \frac{1}{\omega - \mathbf{k}\cdot\mathbf{v} + i\epsilon} &= i \int_0^{2\pi} \frac{d\phi}{2\pi} \int_{-1}^{1} \frac{d\cos(\theta)}{2} \frac{1}{\omega - ik\cos(\theta) + i\epsilon} \\ &= -\frac{1}{2k} \int_{-1}^{1} d\log(\omega - ikx + i\epsilon) \\ &= -\frac{1}{2k} \log\left(\frac{\omega - k + i\epsilon}{\omega + k + i\epsilon}\right) \end{aligned}$$

Kinetic theory, noninteracting





$$\underbrace{p^\mu \partial_\mu f}_{\text{free streaming}} = \underbrace{\Gamma^i_{\alpha\beta} p^\alpha p^\beta \nabla^{(p)}_i f}_{\text{external source}} = S$$

$$G^{00,00} = 3sTi\omega \int \frac{d\Omega}{4\pi} \frac{1}{\omega - \mathbf{v}\cdot\mathbf{k}} = -sT\frac{3\omega}{2k}\log\left(\frac{\omega - k}{\omega + k}\right)$$

… and in a simple interacting kinetic theory ..

Simple toy model: Relaxation Time Approximation

$$\underbrace{p^\mu \partial_\mu f(\mathbf{x}, \mathbf{p}, t)}_{\text{free streaming}} + \underbrace{F^i \nabla_i^{(p)} f(\mathbf{x}, \mathbf{p}, t)}_{\text{external force}} = \frac{p^0}{\tau_R}(f - f_{eq})$$

- Interactions try to make distribution fall on equilibrium f_{eq} in a time scale of τ_R

... linearize and Fourier transfrom ...

$$(-i\omega + i\mathbf{k}\cdot\mathbf{v})\delta f + \frac{1}{p}F^i\nabla_i^{(p)} f = \frac{1}{\tau_R}(\delta f - \delta f_{eq})$$

- δf_{eq}: As the distribution has changed, so have the conserved quantities *locally*. The equilibrium to which the system wants to relax differs at different points in space.

In diffusion channel:

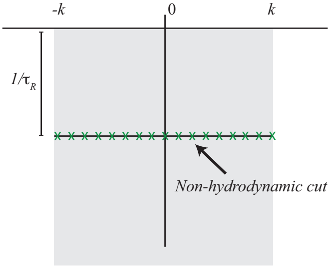
$$f_{eq} = f_{eq}^{g} + \delta f_{eq} = e^{\frac{u\cdot p - \mu}{T}} = e^{-\beta p} + \frac{\delta\mu}{T_0} e^{-\beta p} \ldots$$

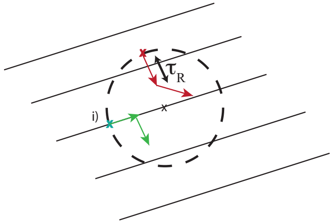
$$n + \delta n = \int_p (f_{eq^g} + \delta f_{eq}) = n_0 + \frac{\delta\mu}{T_0} n_0$$

similarly for δT, δu

Kinetic theory, relaxation time model

(large $t_R k \gg 1$)





As particles get deviated in time τ, the cut gets an imaginary part.

$t_D = \tau_\pi = \tau_R$ sorry for taking picture from different sources...

$$\delta f = -i \frac{F^i \nabla_i^{(p)} f_{eq}^g + \frac{1}{\tau_R} f_{eq} \frac{\delta n}{n}}{-\omega + \mathbf{v} \cdot \mathbf{k} - i/\tau}$$

Romatshke EPJ. C76 (2016)

Appearance of the hydro pole

Because of the particle number conservation δf contains δn which is a function of δf. Solve self consistently:

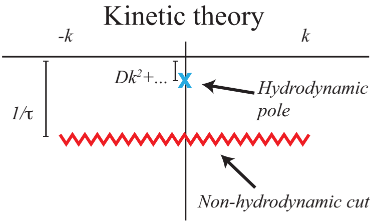
$$\delta n = -i \int \frac{d^3p}{(2\pi)^3} \frac{F^i \nabla_i^{(p)} f_{eq}^g}{-\omega + \mathbf{v}\cdot\mathbf{k} - i/\tau} - i\delta n \int \frac{d^3p}{(2\pi)^3} \frac{\frac{1}{\tau_R} f_{eq} \frac{1}{n}}{-\omega + \mathbf{v}\cdot\mathbf{k} - i/\tau}$$

A new pole appears

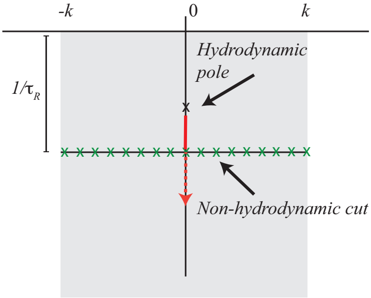
$$\delta n = \frac{-i \int \frac{d^3p}{(2\pi)^3} \frac{F^i \nabla_i^{(p)} f_{eq}^g}{-\omega+\mathbf{v}\cdot\mathbf{k}-i/\tau}}{1 + i \int \frac{d^3p}{(2\pi)^3} \frac{\frac{1}{\tau_R} f_{eq} \frac{1}{n}}{-\omega+\mathbf{v}\cdot\mathbf{k}-i/\tau}}$$

Relaxation time model

(small $\tau_R k \ll 1$, Shear channel for simplicity)



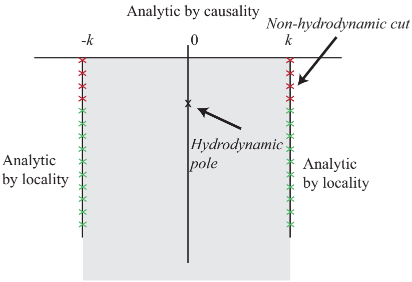
- hydrodynamic mode
- quasi-particle cut



- as k increases, hydro pole drops below cut
- "hydrodynamical transition" at $k \sim 1/t_R$ Romatshke EPJ. C76 (2016)

Hydrodynamical ~~transition~~ fading with k

- Sharp hydrodynamical transition in RTA is consequence of scale-independent relaxation time τ.
- Relaxing this condition, $\tau_R \to \tau_R(p)$, the sharp distinction between a hydro and a non-hydro regime disappears.
- hydro-mode still dominant at small k



Kurkela, Wiedemann, arXiv:1712.04376

End of 2nd Lecture