Introduction to Saturation Physics
and the Spin Puzzle

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Outline

• General concepts
• Classical small-x physics:
  – DIS, classical gluon fields, McLerran-Venugopalan model, parton saturation, saturation scale
  – Glauber-Mueller formula, black disk limit
• Quantum (small-x) evolution:
  – Non-linear BK and JIMWLK evolution equations
  – Solution of BK and JIMWLK equations, energy dependence of the saturation scale, geometric scaling
  – DIS phenomenology at small x and EIC physics
• Proton Spin at Small x:
  – Quark helicity distribution at small x
  – Gluon helicity distribution at small x
  – Their impact on spin puzzle/crisis.
General Concepts
For short distances $x < 0.2$ fm, or, equivalently, large momenta $k > 1$ GeV the QCD coupling is small $\alpha_s \ll 1$ and interactions are weak.
A Question

- Can we understand, qualitatively or even quantitatively, the structure of hadrons and their interactions in High Energy Collisions?
  - What are the total cross sections?
  - What are the multiplicities and production cross sections?
  - Diffractive cross sections.
  - Particle correlations.
What sets the scale of running QCD coupling in high energy collisions?

- “Optimist”: \[ \alpha_s = \alpha_s \left( \sqrt{s} \right) \ll 1 \]

- Pessimist: \[ \alpha_s = \alpha_s \left( \Lambda_{QCD} \right) \sim 1 \] we simply can not tackle high energy scattering in QCD.

- pQCD expert: only study high-\(p_T\) particles such that \[ \alpha_s = \alpha_s \left( p_T \right) \ll 1 \]

But: what about total cross section? bulk of particles?
What sets the scale of running QCD coupling in high energy collisions?

- Saturation physics is based on the existence of a large internal momentum scale $Q_S$ which grows with both energy $s$ and nuclear atomic number $A$

$$Q_S^2 \sim A^{1/3} s^\lambda$$

such that

$$\alpha_s = \alpha_s(Q_S) \ll 1$$

and we can calculate total cross sections, particle spectra and multiplicities, etc, from first principles.
The main principle

- Saturation physics is based on the existence of a large internal transverse momentum scale $Q_s$ which grows with both decreasing Bjorken $x$ and with increasing nuclear atomic number $A$

$$Q_s^2 \sim A^{1/3} \left( \frac{1}{x} \right)^\lambda$$

such that

$$\alpha_s = \alpha_s(Q_s) \ll 1$$

and we can use perturbation theory to calculate total cross sections, particle spectra and multiplicities, correlations, etc, from first principles.
Classical Fields
Deep Inelastic Scattering

• One can prove the structure of a proton by shooting electrons at it at high energies: this is called Deep Inelastic Scattering (DIS).
Deep Inelastic Scattering

Here is a typical deep inelastic positron-proton scattering event.

ZEUS experiment at HERA collider, DESY lab in Hamburg, Germany.
Photon carries 4-momentum $q_\mu$, its virtuality is

$$Q^2 = -q_\mu q^\mu$$

Photon hits a quark in the proton carrying momentum $x_{Bj} P$ with $p$ being the proton’s momentum. Parameter $x_{Bj}$ is the Bjorken $x$ variable.
Physical Meaning of \( Q \)

Uncertainty principle teaches us that

\[ \Delta p \Delta l \approx \hbar \]

which means that the photon probes the proton at the distances of the order \((\hbar=1)\)

\[ \Delta l \sim \frac{1}{Q} \]

Large Momentum \( Q \) = Short Distances Probed
Physical Meaning of Bjorken $x$

The quarks and gluons that interact with the target have their typical momenta on the order of the typical momentum in the target,

$$x_{Bj} \, p \approx q \approx m.$$ 

Then the energy of the collision

$$E \sim p \sim \frac{1}{x_{Bj}}$$

High Energy $=$ Small $x$
Gluons at Small-$x$

- There is a large number of small-$x$ gluons (and quarks) in a proton:

\[ G(x, Q^2), q(x, Q^2) = \text{gluon and quark number densities (q=\(u,d\), or S for sea)} \]

- \( G(x, Q^2), q(x, Q^2) \) = gluon and quark number densities (q=\(u,d\), or S for sea).
Gluons and Quarks in the Proton

⇒ There is a huge number of quarks, anti-quarks and gluons at small-x!

⇒ How do we reconcile this result with the picture of the proton made up of three valence quarks?

⇒ Qualitatively we understand that these extra quarks and gluons are emitted by the original three valence quarks in the proton.
A. McLerran-Venugopalan Model
McLerran-Venugopalan Model

- The wave function of a single nucleus has many small-x quarks and gluons in it.
- In the transverse plane the nucleus is densely packed with gluons and quarks.

**Large occupation number ⇒ Classical Field**
McLerran-Venugopalan Model

- Large gluon density gives a large momentum scale $Q_s$ (the saturation scale): $Q_s^2 \sim \# \text{ gluons per unit transverse area } \sim A^{1/3}$ (nuclear oomph).
- For $Q_s \gg \Lambda_{QCD}$, get a theory at weak coupling $\alpha_s(Q_s^2) \ll 1$ and the leading gluon field is classical.

\[ M = \rho V = \rho \frac{4}{3} \pi R^3 \sim A \Rightarrow R \sim A^{1/3} \]
Small-x gluon "sees" the whole nucleus coherently in the longitudinal direction! It "sees" many color charges which form a net effective color charge $Q = g \text{ (# charges)}^{1/2}$, such that $Q^2 = g^2 \text{ #charges}$ (random walk).

Define color charge density

$$\mu^2 = \frac{Q^2}{S_{\perp}} = \frac{g^2 \text{ # charges}}{S_{\perp}} \propto g^2 \frac{A}{S_{\perp}} \propto A^{1/3}$$

such that for a large nucleus ($A >> 1$)

$$\mu^2 \propto \Lambda_{QCD}^2 A^{1/3} \gg \Lambda_{QCD}^2 \implies \alpha_s(\mu^2) \ll 1$$

Nuclear small-x wave function is perturbative!!!

$$\mu = \frac{Q_s}{20}$$
Saturation Scale

To argue that \( Q_S^2 \sim A^{1/3} \), let us consider an example of a particle scattering on a nucleus. As it travels through the nucleus it bumps into nucleons. Along a straight line trajectory it encounters \( \sim R \sim A^{1/3} \) nucleons, with \( R \) the nuclear radius and \( A \) the atomic number of the nucleus.

The particle receives \( \sim A^{1/3} \) random kicks. Its momentum gets broadened by

\[
\Delta k \sim \sqrt{A^{1/3}} \Rightarrow (\Delta k)^2 \sim A^{1/3}
\]

Saturation scale, as a feature of a collective field of the whole nucleus also scales \( \sim A^{1/3} \).
To find the classical gluon field $A_\mu$ of the nucleus one has to solve the non-linear analogue of Maxwell equations – the Yang-Mills equations, with the nucleus as a source of the color charge:

$$ D_\nu F^{\mu\nu} = J^\mu $$

Yu. K. ’96; J. Jalilian-Marian et al, ‘96
Here’s one of the diagrams showing the non-Abelian gluon field of a large nucleus.

The resummation parameter is $\alpha_s^2 A^{1/3}$, corresponding to two gluons per nucleon approximation.
Classical Gluon Field of a Nucleus

Using the obtained classical gluon field one can construct corresponding gluon distribution function (gluon WW TMD):

\[ \phi_A(x, k^2) \sim \langle A(-k) \cdot A(k) \rangle \]

with the field in the \( A^+=0 \) gauge

\[ \phi_A(x, k_T^2) = \frac{C_F}{\alpha_s \pi} \int \frac{d^2 x_\perp}{x_\perp^2} e^{i k \cdot x} \left[ 1 - \exp \left( -\frac{x_\perp^2 Q_s^2}{4} \ln \frac{1}{x_\perp \Lambda} \right) \right] \]


\( Q_S = \mu \) is the saturation scale

Note that \( \phi \sim \langle A_\mu A_\mu \rangle \sim 1/\alpha \) such that \( A_\mu \sim 1/g \), which is what one would expect for a classical field.
In the UV limit of $k \to \infty$, $x_T$ is small and one obtains

$$
\phi_A(x, k^2_T) = \frac{C_F}{\alpha_s \pi} \int \frac{d^2 x_\perp}{x^2_\perp} e^{i k \cdot x} \left[ 1 - \exp \left( -\frac{x^2_\perp Q_s^2}{4} \ln \frac{1}{x_\perp \Lambda} \right) \right]
$$

which is the usual LO result.

In the IR limit of small $k_T$, $x_T$ is large and we get

$$
\phi_A(x, k^2_T) \sim \int d^2 x_\perp e^{i k \cdot x} Q_s^2 \ln \frac{1}{x_\perp \Lambda} \propto \frac{Q_s^2}{k^2_T}
$$

SATURATION!
Divergence is regularized.
A good object to plot is the classical gluon TMD distribution multiplied by the phase space $k_T$:

Most gluons in the nuclear wave function have transverse momentum of the order of $k_T \sim Q_S$ and $Q_S^2 \sim A^{1/3}$.

We have a small coupling description of the whole wave function in the classical approximation.
B. Glauber-Mueller Rescatterings
In the dipole picture of DIS the virtual photon splits into a quark-antiquark pair, which then interacts with the target.

The total DIS cross section and structure functions are calculated via:
Dipole Amplitude

- The total DIS cross section is expressed in terms of the (Im part of the) forward quark dipole amplitude $N$:

$$
\sigma_{tot}^{\gamma^* A} = \int \frac{d^2 x_\perp}{2 \pi} \int_0^1 \frac{d z}{z (1 - z)} |\Psi^{\gamma^* \rightarrow q\bar{q}}(\vec{x}_\perp, z)|^2 N(\vec{x}_\perp, \vec{b}_\perp, Y)
$$

![Dipole Amplitude Diagram]

with rapidity $Y = \ln(1/x)$
The DIS process in the rest frame of the target is shown below. It factorizes into

\[ \sigma_{\text{tot}}^{\gamma^* A}(x_{Bj}, Q^2) = |\Psi_{\gamma^* \rightarrow q\bar{q}}|^2 \otimes N(x_\perp, Y = \ln 1/x_{Bj}) \]

with rapidity \( Y = \ln(1/x) \)

\( \gamma^* \)

nucleons in the nucleus
Dipole Amplitude

• The quark dipole amplitude is defined by

\[ N(x_1, x_2) = 1 - \frac{1}{N_c} \langle \text{tr} \left[ V(x_1) V^\dagger(x_2) \right] \rangle \]

• Here we use the Wilson lines along the light-cone direction

\[ V(x) = \text{P exp} \left[ i g \int_{-\infty}^{\infty} dx^+ A^-(x^+, x^- = 0, x) \right] \]

• In the classical Glauber-Mueller/McLerran-Venugopalan approach the dipole amplitude resums multiple rescatterings:
Quasi-classical dipole amplitude

A.H. Mueller, ‘90

Lowest-order interaction with each nucleon – two gluon exchange – the same resummation parameter as in the MV model:

\[ \alpha_s^2 A^{1/3} \]
Quasi-classical dipole amplitude

• To resum multiple rescatterings, note that the nucleons are independent of each other and rescatterings on the nucleons are also independent.

• One then writes an equation (Mueller ‘90)

\[
\frac{\partial}{\partial b^+}\tilde{s} = \tilde{s}
\]

Each scattering!

\[
N(x_\perp, Y) = 1 - \exp \left[ -\frac{x_\perp^2 Q_s^2}{4} \ln \frac{1}{x_\perp \Lambda} \right]
\]
DIS in the Classical Approximation

The dipole-nucleus amplitude in the classical approximation is

\[ N(x_\perp, Y) = 1 - \exp \left[ -\frac{x_\perp^2 Q_s^2}{4} \ln \frac{1}{x_\perp \Lambda} \right] \]

A.H. Mueller, '90

Black disk limit,

\[ \sigma_{tot} < 2\pi R^2 \]
Black Disk Limit

- Start with basic scattering theory: the final and initial states are related by the S-matrix operator,

\[ |\psi_f\rangle = \hat{S} |\psi_i\rangle \]

- Write it as

\[ |\psi_f\rangle = |\psi_i\rangle + \left[ \hat{S} - 1 \right] |\psi_i\rangle \]

- The total cross section is

\[ \sigma_{tot} \propto \left| \left[ \hat{S} - 1 \right] |\psi_i\rangle \right|^2 = 2 - S - S^* \]

where the forward matrix element of the S-matrix operator is

\[ S = \langle \psi_i | \hat{S} |\psi_i\rangle \]

and we have used unitarity of the S-matrix

\[ \hat{S} \hat{S}^\dagger = 1 \]
Black Disk Limit

• Now, since \[ |\psi_f\rangle = |\psi_i\rangle + \left[ \hat{S} - 1 \right] |\psi_i\rangle \]
  the elastic cross section is
  \[ \sigma_{el} \propto \left| \langle \psi_i | \left[ \hat{S} - 1 \right] |\psi_i\rangle \right|^2 = |1 - S'|^2 \]

• The inelastic cross section can be found via
  \[ \sigma_{tot} = \sigma_{inel} + \sigma_{el} \]

• In the end, for scattering with impact parameter b we write
  \[ \sigma_{tot} = 2 \int d^2b \left[ 1 - \text{Re} \, S(b) \right] \]
  \[ \sigma_{el} = \int d^2b \left| 1 - S(b) \right|^2 \]
  \[ \sigma_{inel} = \int d^2b \left[ 1 - |S(b)|^2 \right] \]
Unitarity Limit

• Unitarity implies that

\[ 1 = \langle \psi_i | \hat{S} \hat{S}^\dagger | \psi_i \rangle = \sum_X \langle \psi_i | \hat{S} | X \rangle \langle X | \hat{S}^\dagger | \psi_i \rangle \geq |S|^2 \]

• Therefore

\[ |S| \leq 1 \]

leading to the unitarity bound on the total cross section

\[ \sigma_{tot} = 2 \int d^2 b \ [1 - \text{Re} \ S(b)] \leq 4 \int d^2 b = 4\pi R^2 \]

• Notice that when S=-1 the inelastic cross section is zero and

\[ \sigma_{tot} = 2 \int d^2 b \ [1 - \text{Re} \ S(b)] \]

\[ \sigma_{el} = \int d^2 b \ |1 - S(b)|^2 \]

\[ \sigma_{inel} = \int d^2 b \ \left[ 1 - |S(b)|^2 \right] \]

This limit is realized in low-energy scattering!
Black Disk Limit

- At high energy inelastic processes dominate over elastic. Imposing
  \[ \sigma_{inel} \geq \sigma_{el} \]
  we get
  \[ \text{Re } S \geq 0 \]

- The bound on the total cross section is (aka the black disk limit)
  \[ \sigma_{tot} = 2 \int d^2b \left[ 1 - \text{Re } S \right] \leq 2 \int d^2b = 2\pi R^2 \]

- The inelastic and elastic cross sections at the black disk limit are
  \[ \sigma_{inel} = \sigma_{el} = \pi R^2 \]
  \[ \sigma_{tot} = 2 \int d^2b \left[ 1 - \text{Re } S(b) \right] \]
  \[ \sigma_{el} = \int d^2b \left| 1 - S(b) \right|^2 \]
  \[ \sigma_{inel} = \int d^2b \left[ 1 - \left| S(b) \right|^2 \right] \]
Notation

- At high energies

\[ \text{Im } S \approx 0 \]

while the dipole amplitude \( N \) is the imaginary part of the T-matrix \((S=1+iT)\), such that

\[ \text{Re } S = 1 - N \]

- The cross sections are

\[
\begin{align*}
\sigma_{\text{tot}} &= 2 \int d^2 b \, N(x_\perp, b_\perp) \\
\sigma_{\text{el}} &= \int d^2 b \, N^2(x_\perp, b_\perp) \\
\sigma_{\text{inel}} &= \int d^2 b \left[ 2 \, N(x_\perp, b_\perp) - N^2(x_\perp, b_\perp) \right]
\end{align*}
\]

- We see that \( N=1 \) is the black disk limit. Hence \( N \leq 1 \) as we saw above.
The dipole-nucleus amplitude in the classical approximation is

\[ N(x_\perp, Y) = 1 - \exp \left[ -\frac{x_\perp^2 Q_s^2}{4} \ln \frac{1}{x_\perp \Lambda} \right] \]

A.H. Mueller, ‘90

Black disk limit,

\[ \sigma_{tot} < 2\pi R^2 \]
Summary

• We have reviewed the McLerran-Venugopalan model for the small-x wave function of a large nucleus.
• We saw the onset of gluon saturation and the appearance of a large transverse momentum scale – the saturation scale:
  \[ Q_s^2 \sim A^{1/3} \]
• We applied the quasi-classical approach to DIS, obtaining Glauber-Mueller formula for multiple rescatterings of a dipole in a nucleus.
• We saw that onset of saturation ensures that unitarity (the black disk limit) is not violated. Saturation is a consequence of unitarity!
Quantum Small-x Evolution
A. Birds-Eye View
Why Evolve?

• No energy or rapidity dependence in classical field and resulting cross sections.

• Energy/rapidity-dependence comes in via quantum corrections.

• Quantum corrections are included through “evolution equations”.
BFKL Equation

Balitsky, Fadin, Kuraev, Lipatov ‘78

Start with $N$ gluons in the proton’s wave function. As we increase the energy a new gluon can be emitted by either one of the $N$ gluons. The number of newly emitted particles is proportional to $N$.

The BFKL equation for the number of gluons $N$ reads:

$$\frac{\partial}{\partial \ln(1/x)} N(x, Q^2) = \alpha_s K_{BFKL} \otimes N(x, Q^2)$$
As energy increases BFKL evolution produces more gluons, roughly of the same size. The gluons overlap each other creating areas of very high density.

Number density of gluons, along with corresponding cross sections grows as a power of $1/x$ or, equivalently, of energy ($s$)

$$N \sim e^{\Delta \ln(1/x)} = \left(\frac{1}{x}\right)^{\frac{\Delta}{x}} \sim s^{\Delta}$$
But can parton densities rise forever? Can gluon fields be infinitely strong? Can the cross sections rise forever?

No! There exists a black disk limit for cross sections, which we know from Quantum Mechanics: for high-energy scattering on a disk of radius $R$ the total cross section is bounded by

$$\sigma_{tot} \leq 2 \pi R^2$$
Nonlinear Equation

At very high energy gluon recombination becomes important. As energy (rapidity) increases, gluons not only split into more gluons, but also recombine. Recombination reduces the number of gluons in the wave function. Here \( Y \sim \ln s \sim \ln \frac{1}{x} \) is rapidity, \( s \) is cms energy squared.

\[
\frac{\partial}{\partial Y} N(x, k_T^2) = \alpha_s K_{BFKL} \otimes N(x, k_T^2) - \alpha_s [N(x, k_T^2)]^2
\]

Number of gluon pairs \( \sim N^2 \)

I. Balitsky ’96, Yu. K. ’99 (large \( N_c \))

\[
Y = \ln \frac{1}{x}
\]
Gluon recombination tries to reduce the number of gluons in the wave function. At very high energy recombination begins to compensate gluon splitting. Gluon density reaches a limit and does not grow anymore. Ditto for the total DIS cross sections. Black disk limit and unitarity are restored!
B. In-Depth Discussion
Quantum Evolution

As energy increases the higher Fock states including gluons on top of the quark-antiquark pair become important. They generate a cascade of gluons.

These extra gluons bring in powers of $\alpha_s \ln s$, such that when $\alpha_s \ll 1$ and $\ln s \gg 1$ this parameter is $\alpha_s \ln s \sim 1$ (leading logarithmic approximation, LLA).
In the large-$N_C$ limit of QCD the gluon corrections become color dipoles. Gluon cascade becomes a dipole cascade.

A. H. Mueller, ’93–’94

We need to resum dipole cascade, with each final state dipole interacting with the target.

Yu. K. ‘99
Notation (Large-$N_C$)

Real emissions in the amplitude squared
(dashed line – all Glauber-Mueller exchanges at light-cone time = 0)

Virtual corrections in the amplitude (wave function)
Nonlinear Evolution

To sum up the gluon cascade at large-N_{c} we write the following equation for the dipole S-matrix:

To sum up the gluon cascade at large-N_{c} we write the following equation for the dipole S-matrix:

\[ \frac{\partial}{\partial Y} S_{x_0, x_1}(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 x_2 \frac{x_{01}^2}{x_{02} x_{21}^2} \left[ S_{x_0, x_2}(Y) S_{x_2, x_1}(Y) - S_{x_0, x_1}(Y) \right] \]

Remembering that S= 1-N we can rewrite this equation in terms of the dipole scattering amplitude N.
Nonlinear evolution at large $N_c$

As $N=1$-S we write

\[ \frac{\partial}{\partial Y} N_{x_0,x_1}(Y) = \frac{\alpha_s N_c}{2 \pi^2} \int d^2 x_2 \frac{x_{01}^2}{x_{02} x_{21}^2} \left[ N_{x_0,x_2}(Y) + N_{x_2,x_1}(Y) - N_{x_0,x_1}(Y) - N_{x_0,x_2}(Y) N_{x_2,x_1}(Y) \right] \]

Balitsky '96, Yu.K. '99
Nonlinear Evolution Equation

\[
\frac{\delta}{\delta \ln s} N(\mathbf{x}_0, Y) = 2
\]

We can resum the dipole cascade

\[
\frac{\partial N(x_{01}, Y)}{\partial Y} = \frac{\alpha_s N_C}{\pi^2} \int d^2 x_2 \left[ \frac{x_{01}^2}{x_{02} x_{12}} - 2\pi \delta^2(x_{01} - x_{02}) \ln \left( \frac{x_{01}}{\rho} \right) \right] N(x_{02}, Y)
\]

\[
- \frac{\alpha_s N_C}{2\pi^2} \int d^2 x_2 \frac{x_{01}^2}{x_{02} x_{12}} N(x_{02}, Y) N(x_{12}, Y)
\]

\[
N(x_{\perp}, Y) = 1 - \exp \left[ -\frac{x_{\perp}^2 Q_s^2}{4} \ln \frac{1}{x_{\perp} \Lambda} \right]
\]

I. Balitsky, ’96, HE effective lagrangian
Yu. K., ’99, large \( N_C \) QCD

\( \Rightarrow \) Linear part is BFKL, quadratic term brings in damping

initial condition
Resummation parameter

- BK equation resums powers of
  \[ \alpha_s \, N_c \, Y \]

- The Galuber-Mueller/McLerran-Venugopalan initial conditions for it resum powers of
  \[ \alpha_s^2 \, A^{1/3} \]
Going Beyond Large $N_C$: JIMWLK

To do calculations beyond the large-$N_C$ limit on has to use a functional integro-differential equation written by Iancu, Jalilian-Marian, Kovner, Leonidov, McLerran and Weigert (JIMWLK):

$$\frac{\partial Z}{\partial Y} = \alpha_s \left\{ \frac{1}{2} \frac{\delta^2}{\delta \rho(u) \delta \rho(v)} [Z \chi(u, v)] - \frac{\delta}{\delta \rho(u)} [Z \sigma(u)] \right\}$$

where the functional $Z[\rho]$ can then be used for obtaining wave function-averaged observables (like Wilson loops for DIS):

$$\langle O \rangle = \int D\rho \ Z[\rho] \ O[\rho]$$
JIMWLK: derivation outline

A.H. Mueller, 2001

- Start by introducing a weight functional, $W_Y[\alpha]$. Here $\alpha = A^+$ is the gluon field of the target proton or nucleus. $\alpha(x^-, \bar{x}) \equiv A^+(x^+ = 0, x^-, \bar{x})$
- The functional is used to generate expectation values of gluon-field dependent operators in the target state:

$$\langle \hat{O}_\alpha \rangle_Y = \int \mathcal{D}\alpha \ \hat{O}_\alpha \ W_Y[\alpha]$$

- Imagine that we know small-x evolution for some operator $O$:

$$\partial_Y \langle \hat{O}_\alpha \rangle_Y = \langle \mathcal{K}_\alpha \otimes \hat{O}_\alpha \rangle_Y = \int \mathcal{D}\alpha \ [\mathcal{K}_\alpha \otimes \hat{O}_\alpha] \ W_Y[\alpha]$$

- On the other hand, we can differentiate the first equation above,

$$\partial_Y \langle \hat{O}_\alpha \rangle_Y = \int \mathcal{D}\alpha \ \hat{O}_\alpha \ \partial_Y W_Y[\alpha]$$

- Comparing the last two equations, and integrating by parts in the second to last equation, we will arrive at and equation for the weight functional $W_Y[\alpha]$. 
JIMWLK: derivation outline

- As a test operator, take a pair of Wilson lines (not a dipole!):

\[ \hat{O}_{\vec{x}_1 \perp, \vec{x}_0 \perp} = V_{\vec{x}_1 \perp} \otimes V_{\vec{x}_0 \perp}^\dagger \]

- Construct the evolution of this operator by summing the following familiar diagrams:
In the end one arrive at the JIMWLK evolution equation (1997-2002):

\[
\partial_Y W_Y[\alpha] = \alpha_s \left\{ \frac{1}{2} \int d^2 x_\perp d^2 y_\perp \frac{\delta^2}{\delta \alpha^a(x^- , \vec{x}_\perp) \delta \alpha^b(y^- , \vec{y}_\perp)} \left[ \eta_{\vec{x}_\perp \vec{y}_\perp}^{ab} W_Y[\alpha] \right] \\
- \int d^2 x_\perp \frac{\delta}{\delta \alpha^a(x^- , \vec{x}_\perp)} \left[ \nu_{\vec{x}_\perp}^a W_Y[\alpha] \right] \right\}
\]

with

\[
\eta_{\vec{x}_1 \perp \vec{x}_0 \perp}^{ab} = \frac{4}{g^2 \pi^2} \int d^2 x_2 \frac{\vec{x}_{21} \cdot \vec{x}_{20}}{x_{21}^2 x_{20}^2} \left[ 1 - U_{\vec{x}_1 \perp} U_{\vec{x}_2 \perp}^\dagger - U_{\vec{x}_2 \perp} U_{\vec{x}_1 \perp}^\dagger + U_{\vec{x}_1 \perp} U_{\vec{x}_0 \perp}^\dagger \right]^{ab}
\]

\[
\nu_{\vec{x}_1 \perp}^a = \frac{i}{g \pi^2} \int d^2 x_2 \frac{\vec{x}_{21}}{x_{21}^2} \text{Tr} \left[ T^a U_{\vec{x}_1 \perp} U_{\vec{x}_2 \perp}^\dagger \right]
\]

- Here U is the adjoint Wilson line on a light cone,

\[
U_{\vec{x}_\perp} = \text{P exp} \left\{ i g \int_{-\infty}^{\infty} dx^- A^+(x^+ = 0, x^-, \vec{x}_\perp) \right\}
\]
JIMWLK Equation

- JIMWLK equation can be used to construct any-\(N_C\) small-x evolution of any operator made of infinite light-cone Wilson lines (in any representation), such as color-dipole, color-quadrupole, etc., and other operators.

- Since

\[
\Box \alpha(x^-, \bar{x}) = \rho(x^-, \bar{x})
\]

JIMWLK evolution can be re-written in terms of the color density \(\rho\) in the kernel.

- JIMWLK approach sums up powers of \(\alpha_s Y\) and \(\alpha_s^2 A^{1/3}\)
Solving JIMWLK

• The JIMWLK equation was solved on the lattice by K. Rummukainen and H. Weigert ’04 (and others since).

• For the dipole amplitude $N(x_0, x_1, Y)$, the relative corrections to the large-$N_C$ limit BK equation are $< 0.001$ ! Not the naïve $1/N_C^2 \sim 0.1$ ! (For realistic rapidities/energies.)

• The reason for that is dynamical, and is largely due to saturation effects suppressing the bulk of the potential $1/N_C^2$ corrections (Yu.K., J. Kuokkanen, K. Rummukainen, H. Weigert, ‘08).
Last time

• We discussed the McLerran-Venugopalan (MV) model: classical gluon field of a nucleus.
• Found the classical gluon distribution.
• Argued that the saturation scale grows as
\[ Q_s^2 \sim A^{1/3} \]
• Considered DIS in the quasi-classical picture:
Last time

• Derived the nonlinear (BK) evolution equation:

\[ \frac{\partial}{\partial N} Y \]  

\[ \ln x \]

• Resummation parameter is (leading log approximation):

\[ \alpha_s Y = \alpha_s \ln \frac{1}{x} \sim \alpha_s \ln s \]
Last time

- The equation reads:

\[
\partial_Y N_{x_0,x_1}(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 x_2 \frac{x_{01}^2}{x_0^2 x_1^2} \left[ N_{x_0,x_2}(Y) + N_{x_2,x_1}(Y) - N_{x_0,x_1}(Y) - N_{x_0,x_2}(Y) N_{x_2,x_1}(Y) \right]
\]

- It combines BFKL evolution (the linear part) and the quadratic damping correction.
- All-$N_c$ evolution is JIMWLK.
- Now let’s discuss solution of BK evolution.
C. Solution of BK Equation
Solution of BK equation

BK solution preserves the black disk limit, $N<1$ always (unlike the linear BFKL equation)

\[
\sigma_{q\bar{q}A} = 2 \int d^2b \, N(x_\perp, b_\perp, Y)
\]

numerical solution by J. Albacete ‘03 (earlier solutions were found numerically by Golec-Biernat, Motyka, Stasto, by Braun and by Lublinsky et al in ‘01)
Saturation scale

\[ Q_s(Y) \text{ (GeV)} \]

numerical solution by J. Albacete
BK Solution

- Preserves the black disk limit, $N<1$ always.

\[ \sigma^{q\bar{q}A} = 2 \int d^2b \, N(x_\perp, b_\perp, Y) \]

- Avoids the IR problem of BFKL evolution due to the saturation scale screening the IR:

Golec-Biernat, Motyka, Stasto ‘02
In the conventional Feynman diagram picture the BFKL equation can be represented by a ladder graph shown here. Each rung of the ladder brings in a power of $\alpha \ln s$.

The resulting dipole amplitude grows as a power of energy

$$N \sim s^\Delta$$

violating Froissart unitarity bound

$$\sigma_{tot} \leq \text{const} \ln^2 s$$
Gribov, Levin and Ryskin ('81) proposed summing up “fan” diagrams:

Mueller and Qiu ('85) summed “fan” diagrams for large $Q^2$.

The GLR-MQ equation reads:

$$\frac{\partial}{\partial \ln 1/x} \phi(x, k_T^2) = \alpha_s K_{BFKL} \otimes \phi(x, k_T^2) - \alpha_s [\phi(x, k_T^2)]^2$$

GLR-MQ equation has the same principle of recombination as BK and JIMWLK. GLR-MQ equation was thought about as the first nonlinear correction to the linear BFKL evolution. An AGL (Ayala, Gay Ducati, Levin '96) equation was suggested to resum higher-order nonlinear corrections.

BK/JIMWLK derivation showed that for the dipole amplitude $N$ (!) there are no more terms in the large-$N_C$ limit and obtained the correct kernel for the nonlinear term (compared to GLR suggestion).
Energy Dependence of the Saturation Scale

Single BFKL ladder gives scattering amplitude of the order

\[ N \sim \frac{\Lambda}{k_T} s^\Delta \]

Nonlinear saturation effects become important when \( N \sim N^2 \Rightarrow N \sim 1 \). This happens at

\[ k_T = Q_s \sim \Lambda s^\Delta \]

Saturation scale grows with energy!

Typical partons in the wave function have \( k_T \sim Q_s \), so that their characteristic size is of the order \( r \sim 1/k_T \sim 1/Q_s \). 

\( \Rightarrow \) Typical parton size decreases with energy!
Saturation scale

\[ Q_s(Y) \text{ (GeV)} \]

\[ \alpha_s Y \]

Numerical solution by J. Albacete
High Density of Gluons

- High number of gluons populates the transverse extent of the proton or nucleus, leading to a very dense saturated wave function known as the Color Glass Condensate (CGC):

Low Energy

Proton \((x_0, Q^2)\)

parton

\(x_0 \gg x\)

High Energy

many new smaller partons are produced

Proton \((x, Q^2)\)

“Color Glass Condensate”
Map of High Energy QCD

energy

resolution, $\ln Q^2$  number of partons

energy, $Y = \ln\frac{1}{x}$  number & density

size of gluons
Map of High Energy QCD

\[ Y = \ln \frac{1}{x} \]

\[ Q_s^2(Y) \sim \left( \frac{1}{x} \right)^\lambda \]

Saturation Scale grows with energy

BFKL, DGLAP – linear equations
BK/JIMWLK – nonlinear

\[ Q^2 \sim \alpha_s \ll 1 \]

\[ \Lambda_{QCD}^2 \]

size of gluons
First partons are produced overlapping each other, all of them about the same size.

When some critical density is reached no more partons of given size can fit in the wave function. The proton starts producing smaller partons to fit them in.
Saturation physics allows us to study regions of high parton density in the small coupling regime, where calculations are still under control!

Transition to saturation region is characterized by the saturation scale

\[ Q_s^2 \sim A^{1/3} \left( \frac{1}{x} \right)^{\lambda} \]
Geometric Scaling

One of the predictions of the JIMWLK/BK evolution equations is geometric scaling:

DIS cross section should be a function of one parameter:

\[ \sigma_{DIS} (x, Q^2) = \sigma_{DIS} \left( \frac{Q^2}{Q_s^2(x)} \right) \]

(Levin, Tuchin ’99; Iancu, Itakura, McLerran ’02)
Geometric Scaling

\[ N(\tau = x_{\perp}Q_s(Y)) \]
\[ \alpha_s Y = 0, 1.2, 2.4, 3.6, 4.8 \]

numerical solution by J. Albacete
Geometric Scaling in DIS

Geometric scaling has been observed in DIS data by Stasto, Golec-Biernat, Kwiecinski in ’00.

Here they plot the total DIS cross section, which is a function of 2 variables - $Q^2$ and $x$, as a function of just one variable:

$$\tau = \frac{Q^2}{Q_s^2}$$
Map of High Energy QCD

Saturation region
Color Glass Condensate

Extended Geometric Scaling region

\[ k_{\text{geom}} \sim \frac{Q_S^2}{Q_{S0}} \]

non-perturbative region
\[ \alpha_s \sim 1 \]

\[ \Lambda_{\text{QCD}}^2 \]
\[ \alpha_s \ll 1 \]

\[ Q^2 \]
Saturation Scale

To summarize, saturation scale is an increasing function of both energy \((1/x)\) and \(A\):

\[ Q_s^2 \sim \left( \frac{A}{x} \right)^{1/3} \]

Gold nucleus provides an enhancement by \(197^{1/3}\), which is equivalent to doing scattering on a proton at 197 times smaller \(x\) / higher \(s\)!
References

• H.Weigert, hep-ph/0501087
• J.Jalilian-Marian, Yu.K., hep-ph/0505052
• and...
References

Quantum Chromodynamics at High Energy

YURI V. KOVCHEGOV
AND EUGENE LEVIN

CAMBRIDGE MONOGRAPHS ON PARTICLE PHYSICS, NUCLEAR PHYSICS AND COSMOLOGY

Published in September 2012 by Cambridge U Press
Conclusions

• We have constructed nuclear/hadronic wave function in the quasi-classical approximation (MV model), and studied DIS in the same approximation.

• We included small-\(x\) evolution corrections into the DIS process, obtaining nonlinear BK/JIMWLK evolution equations.

• We found the saturation scale justifying the whole procedure.

• Saturation/CGC physics predicts geometric scaling observed experimentally at HERA.
D. DIS at Small $x$ Phenomenology
Three-step prescription

• Calculate the observable in the classical approximation.

• Include nonlinear small-x evolution corrections (BK/JIMWLK), introducing energy-dependence.

• To compare with experiment, need to fix the scale of the running coupling.

• NLO corrections to BK/JIMWLK need to be included as well.
Geometric Scaling in DIS

Geometric scaling has been observed in DIS data by Stasto, Golec-Biernat, Kwiecinski in ’00.

Here they plot the total DIS cross section, which is a function of 2 variables - $Q^2$ and $x$, as a function of just one variable:

$$\tau = \frac{Q^2}{Q^2_s}$$
Comparison of rcBK with HERA $F_2$&$F_L$ Data

DIS structure functions:

$$F_2, L = \frac{Q^2}{4 \pi^2 \alpha_E M} \sigma_{\gamma^* p}^{\gamma^* p, \gamma^* p, \gamma^* p, \gamma^* p, \gamma^* p}$$

from Albacete, Armesto, Milhano, Salgado '09

![Graph showing comparison of $F_L$ and $F_2$]
Comparison with the combined H1 and ZEUS data

Albacete, Armesto, Milhano, Qiuroga Arias, and Salgado ‘11

reduced cross section:

\[ \sigma_r = F_2 - \frac{y^2}{1 + (1 - y)^2} F_L \]
Diffractive cross section

Also agrees with the saturation/CGC expectations.
E. A case for EIC
Electron-Ion Collider (EIC) White Paper

- EIC WP was finished in late 2012
- A several-year effort by a 19-member committee + 58 co-authors
- arXiv:1212.1701 [nucl-ex]
- EIC can be realized as eRHIC (BNL) or as ELIC (JLab)
Can Saturation Discovery be Completed at EIC?

EIC has an unprecedented small-x reach for DIS on large nuclear targets, allowing to seal the discovery of saturation physics and study of its properties:
Diffraction on a black disk

- For low $Q^2$ (large dipole sizes) the black disk limit is reached with $N=1$
- Diffraction (elastic scattering) becomes a half of the total cross section

$$\frac{\sigma_{q\bar{q}A}^{el}}{\sigma_{q\bar{q}A}^{tot}} = \frac{\int d^2b \; N^2}{2 \int d^2b \; N} \rightarrow \frac{1}{2}$$

- Large fraction of diffractive events in DIS is a signature of reaching the black disk limit!
Diffractive over total cross sections

- Here’s an EIC stage-I measurement which may distinguish saturation from non-saturation approaches:

\[
\frac{1}{\sigma_{\text{tot}}} \frac{d\sigma_{\text{diff}}}{dM_x^2} (\text{GeV}^{-2})
\]

\[
\text{ratio (eAu/ep)}
\]

\[
M_x^2 (\text{GeV}^2)
\]

sat = Kowalski et al ‘08, plots generated by Marquet
no-sat = Leading Twist Shadowing (LTS), Frankfurt, Guzey, Strikman ‘04, plots by Guzey
Spin at Small-x
Our understanding of nucleon spin structure has evolved:

- In the 1980’s the proton spin was thought of as a sum of constituent quark spins (left panel)
- Currently we believe that the proton spin is a sum of the spins of valence and sea quarks and of gluons, along with the orbital angular momenta of quarks and gluons (right panel)
Helicity Distributions

• To quantify the contributions of quarks and gluons to the proton spin on defines helicity distribution functions: number of quarks/gluons with spin parallel to the proton momentum minus the number of quarks/gluons with the spin opposite to the proton momentum:

\[
\Delta f(x, Q^2) \equiv f^+(x, Q^2) - f^-(x, Q^2)
\]

• The helicity parton distributions are

\[
\Delta \Sigma \equiv \Delta u + \Delta \bar{u} + \Delta d + \Delta \bar{d} + \Delta s + \Delta \bar{s}
\]

with the net quark helicity distribution

and \( \Delta G(x, Q^2) \) the gluon helicity distribution.
Proton Helicity Sum Rule

- Helicity sum rule:

\[ \frac{1}{2} = S_q + L_q + S_g + L_g \]

with the net quark and gluon spin

\[ S_q(Q^2) = \frac{1}{2} \int_0^1 dx \Delta \Sigma(x, Q^2) \quad S_g(Q^2) = \int_0^1 dx \Delta G(x, Q^2) \]

- \( L_q \) and \( L_g \) are the quark and gluon orbital angular momenta
Proton Spin Puzzle

- The spin puzzle began when the EMC collaboration measured the proton g₁ structure function ca 1988. Their data resulted in

\[ \Delta \Sigma \approx 0.1 \div 0.2 \]

- It appeared quarks do not carry all of the proton spin (which would have corresponded to \( \Delta \Sigma = 1 \)).

- Missing spin can be
  - Carried by gluons
  - In the orbital angular momenta of quarks and gluons
  - At small x:

\[
\frac{1}{2} = S_q + L_q + S_g + L_g
\]

\[
S_q(Q^2) = \frac{1}{2} \int_0^1 dx \Delta \Sigma(x, Q^2) \quad S_g(Q^2) = \int_0^1 dx \Delta G(x, Q^2)
\]

Can’t integrate down to zero, use \( x_{\text{min}} \) instead!

- Or all of the above!
Current Knowledge of Proton Spin

- The proton spin carried by the quarks is estimated to be (for $0.001 < x < 1$)
  \[ S_q(Q^2 = 10 \text{ GeV}^2) \approx 0.15 \div 0.20 \]

- The proton spin carried by the gluons is (for $0.05 < x < 1$)
  \[ S_G(Q^2 = 10 \text{ GeV}^2) \approx 0.13 \div 0.26 \]

- Unfortunately the uncertainties are large. Note also that the $x$-ranges are limited, with more spin (positive or negative) possible at small $x$. 
How much spin is at small $x$?

- Uncertainties are very large at small $x$! (EIC may reduce them.)
Spin at small $x$

- The goal of this work is to provide theoretical understanding of helicity PDFs at very small $x$.

- Our work would provide guidance for future hPDFs parametrizations of the existing and new data (e.g., the data to be collected at EIC).

- Alternatively the data can be analyzed using our small-$x$ evolution formalism.
Helicity Evolution at Small x

- To understand how much of the proton’s spin is at small x one can construct a helicity analogue of the BFKL equation:

This new helicity evolution equation is a bit more subtle, since it has to keep track of both quark and gluon helicities. (BFKL/BK/JIMWLK only have gluons at leading order.)

A. Quark Helicity at Small x
(flavor-singlet case)

Observables

- We want to calculate quark helicity PDF and TMD at small $x$.

### Leading Twist TMDs

<table>
<thead>
<tr>
<th>Nucleon Polarization</th>
<th>Quark Polarization</th>
<th>Un-Polarized (U)</th>
<th>Longitudinally Polarized (L)</th>
<th>Transversely Polarized (T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>$f_1 = \bigcirc$</td>
<td></td>
<td></td>
<td>$h_{1L}^{\perp} = \bigcirc - \bigcirc$ Boer-Mulders</td>
</tr>
<tr>
<td>L</td>
<td>$g_{1L} = \bigcirc \rightarrow - \bigcirc$ Helicity</td>
<td></td>
<td>$h_{1L}^{\perp} = \bigcirc - \bigcirc$</td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>$f_{1T}^{\perp} = \bigcirc - \bigcirc$ Sivers</td>
<td>$g_{1T}^{\perp} = \bigcirc - \bigcirc$</td>
<td>$h_{1T} = \bigcirc - \bigcirc$ Transversity</td>
<td>$h_{1T}^{\perp} = \bigcirc - \bigcirc$</td>
</tr>
</tbody>
</table>
Quark Helicity TMD

• We start with the definition of the quark helicity TMD with a future-pointing Wilson line staple.

\[
g_{1L}^q(x, k_T^2) = \frac{1}{(2\pi)^3} \frac{1}{2} \sum_{SL} \int d^2r \, dr^- \, e^{ik\cdot r} \langle p, SL | \bar{\psi}(0) U[0, r] \frac{\gamma^+ \gamma^5}{2} \psi(r) | p, SL \rangle_{r^+=0}
\]

• At small-\(x\), in anticipation of the shock-wave formalism, we rewrite the quark helicity TMD as (in \(A^-=0\) gauge for the + moving proton)

\[
g_{1L}^q(x, k_T^2) = \frac{2p^+}{(2\pi)^3} \int d^2\zeta \, d\zeta^- \, d^2\xi \, d\xi^- \, e^{ik\cdot(\zeta^- - \xi^-)} \left( \frac{1}{2} \gamma^+ \gamma^5 \right)_{\alpha\beta} \langle \bar{\psi}_\alpha(\xi^-) V_{\xi^-}[\xi^-, \infty] \, V_{\xi^-}[\infty, \zeta^-] \psi_\beta(\zeta^-) \rangle
\]

where the fundamental light-cone Wilson line is

\[
V_{\bar{x}}[b^-, a^-] = P \exp \left\{ ig \int_{a^-}^{b^-} dx^- A^+(x^-, \bar{x}) \right\}
\]
Quark Helicity TMD at Small $x$

- At high energy/small-$x$ the proton is a shock wave, and we have the following contributions to the SIDIS quark helicity TMD:

$$g_{1L}^q(x, k_T^2) = \frac{2p^+}{(2\pi)^3} \sum_X \int d^2\zeta \ d\zeta^- \ d^2\xi \ d\xi^- \ e^{ik \cdot (\zeta^- - \xi^-)} \left( \frac{1}{2} \gamma^+ \gamma^5 \right)_{\alpha\beta} \left\langle \overline{\psi}_\alpha(\xi) \ V_{\xi} & [\xi^-, \infty] \ |X\rangle \ \langle X| \ V_{\xi} & [\infty, \zeta^-] \ \psi_\beta(\zeta) \right\rangle$$
Quark Helicity TMD at Small $x$

- Diagram D does not transfer spin information from the target. Diagram C is canceled as we move t-channel quarks across the cut.
- Diagram F is energy-suppressed, since the gluon should have no time to be emitted and absorbed inside the shock wave.
- Diagrams of the types A and E++ can be shown to cancel each other at the leading (DLA) order (Ward identity).
- We are left with the diagram B.
Quark Helicity TMD at Small x

• Dominance of diagram B can also be obtained by applying crossing symmetry to the SIDIS process (KS ‘15):

\[
\begin{align*}
\text{Diagram B} & \quad + \quad \text{Diagram B} \quad + \quad \text{c.c.} \quad = 0 \\
\text{Diagram B} & \quad + \quad \text{Diagram B} \quad + \quad \text{c.c.} \quad = \text{the answer}
\end{align*}
\]

• Compare the last line to the diagram B: reflecting the cc amplitude into the amplitude reduces the above diagram to the one on the right.
Quark Helicity TMD at Small $x$

• Evaluating diagram B we arrive at

$$g_{1L}^{q}(x, k_{T}^{2}) = \frac{4N_{c}}{(2\pi)^{6}} \int d^{2}\zeta d^{2}w d^{2}y e^{-ik\cdot(\zeta-y)} \int_{\Lambda^{2}/s}^{1} \frac{dz}{z} \frac{\zeta - w}{|\zeta - w|^{2}} \cdot \frac{y - w}{|y - w|^{2}} G_{w,\zeta}(zs)$$

where $G_{w,\zeta}$ is the polarized dipole amplitude (defined on the next slide).

• Here $s$ is the cms energy squared, $\Lambda$ is some IR cutoff, underlining denotes transverse vectors, $z$ = smallest longitudinal momentum fraction of the dipole momentum out of those carried by the quark and the antiquark

• The same result was previously obtained starting with the SIDIS process (KPS ‘15) instead of the operator definition of quark helicity TMD: we have thus shown that the two approaches are consistent at small $x$. 
Polarized Dipole

- All flavor-singlet small-x helicity observables depend on one object, "polarized dipole amplitude":

\[ G_{10}(z) \equiv \frac{1}{2N_c} \text{Re} \left\{ T \text{tr} \left[ V_0 V_1^{pol\dagger} \right] + T \text{tr} \left[ V_1^{pol} V_0^{\dagger} \right] \right\}(z) \]

\[ V_x \equiv \mathcal{P} \exp \left[ ig \int_{-\infty}^{\infty} dx^+ A^-(x^+, 0^-, x) \right] \]

- Double brackets denote an object with energy suppression scaled out:

\[ \left\langle \mathcal{O} \right\rangle(z) \equiv zs \left\langle \mathcal{O} \right\rangle(z) \]
To complete the definition of the polarized dipole amplitude, we need to construct the definition of the polarized “Wilson line” $V_{\text{pol}}$, which is the leading helicity-dependent contribution for the quark scattering amplitude on a longitudinally-polarized target proton.

At the leading order we can either exchange one non-eikonal t-channel gluon (with quark-gluon vertices denoted by blobs above) to transfer polarization between the projectile and the target, or two t-channel quarks, as shown above.
Polarized fundamental “Wilson line”

- In the end one arrives at (cf. Chirilli ‘18)

\[
V_{\bar{x}}^{pol} = \frac{ig p_{1}^{+}}{s} \int_{-\infty}^{\infty} dx^{-} V_{\bar{x}}[+\infty, x^{-}] F^{12}(x^{-}, \bar{x}) V_{\bar{x}}[x^{-}, -\infty]
\]

\[
- \frac{g^{2} p_{1}^{+}}{s} \int_{-\infty}^{\infty} dx_{1}^{-} \int_{x_{1}^{-}}^{\infty} dx_{2}^{-} V_{\bar{x}}[+\infty, x_{2}^{-}] t^{b} \psi_{b}(x_{2}^{-}, \bar{x}) U_{x}^{ba}[x_{2}^{-}, x_{1}^{-}] \left[ \frac{1}{2} \gamma^{+} \gamma^{5} \right]_{\alpha\beta} \bar{\psi}_{\alpha}(x_{1}^{-}, \bar{x}) t^{a} V_{\bar{x}}[x_{1}^{-}, -\infty].
\]

- The first term on the right (the gluon exchange contribution) was known before (KPS ‘17), the second term (quark exchange) is new.

- We have employed an adjoint light-cone Wilson line

\[
U_{\bar{x}}[b^{-}, a^{-}] = \mathcal{P} \exp \left[ ig \int_{a^{-}}^{b^{-}} dx^{-} A^{+}(x^{+} = 0, x^{-}, \bar{x}) \right]
\]
Polarized Dipole Amplitude

• The polarized dipole amplitude is then defined by

\[ G_{10}(z) = \frac{1}{4N_c} \int_{-\infty}^{\infty} dx^- \left\langle \text{tr} \left[ V_0[\infty, -\infty] V_1[-\infty, x^-] (-ig) \nabla \times \vec{A}(x^-, x) V_1[x^-, \infty] \right] + \text{c.c.} \right\rangle (z) \]

with the standard light-cone Wilson line

\[ V_x[b^-, a^-] = \text{P exp} \left\{ ig \int_{a^-}^{b^-} dx^- A^+(x^-, x) \right\} \]
Polarized adjoint “Wilson line”

• Quarks mix with gluons. Therefore, we need to construct the adjoint polarized Wilson line --- the leading helicity-dependent part of the gluon scattering amplitude on the longitudinally polarized target.

\[
(U_{\text{pol}}^x)^{ab} = \frac{2i g p_1^+}{s} \int_{-\infty}^{+\infty} dx^- \left( U_x[+\infty, x^-] \mathcal{F}^{12}(x^+ = 0, x^-, x) U_x[x^-, -\infty] \right)^{ab}
\]

\[
- \frac{g^2 p_1^+}{s} \int_{-\infty}^{\infty} dx_2^- \int_{x_1^-}^{+\infty} \psi(x_2^-, x) t^{a'} V_x[x_2^-, x_1^-] \frac{1}{2} \gamma^+ \gamma_5 t^{b'} \psi(x_1^-, x) U_x^{b'b}[x_1^-, -\infty] - \text{c.c.}
\]
Small-x Evolution at large $N_c$

- At large $N_c$ the evolution is gluon-driven. We will evolve a gluon dipole, remembering that at large $N_c$ the relation between the adjoint and fundamental longitudinally-polarized gluon dipoles is

$$G_{10}^{adj}(z) = 4 G_{10}(z)$$

(Note that the factor is 4, not 2 like in the unpolarized dipole case.)
Evolution for Polarized Quark Dipole

One can construct an evolution equation for the polarized dipole:

\[
\frac{\partial}{\partial y} Y_{100} = Y(100) + Y(010) + Y(001) + \text{box} = \text{target shock wave}
\]

Spin-dependent (non-eikonal) vertex

similar to unpolarized BK evolution

polarized particle
Evolution for Polarized Quark Dipole

\[
\frac{1}{N_c} \left\langle \text{tr} \left[ V_0^{\text{unp}} V_1^{\text{pol} \dagger} \right] \right\rangle (z) = \frac{1}{N_c} \left\langle \text{tr} \left[ V_0^{\text{unp}} V_1^{\text{pol} \dagger} \right] \right\rangle_0 (z) + \frac{\alpha_s}{2\pi^2} \int_{z_i}^{z} \frac{dz'}{z'} \int_{\rho'^2} d^2 x_2 \frac{d^2 x_2}{x_{21}^2} \\
\times \left\{ \theta(x_{10} - x_{21}) \frac{2}{N_c} \left\langle \text{tr} \left[ t^b V_0^{\text{unp}} t^a V_1^{\text{unp} \dagger} \right] U_2^{\text{pol} \, ba} \right\rangle (z') \right. \\
+ \left. \theta(x_{10}^2 z - x_{21}^2 z') \frac{1}{N_c} \left\langle \text{tr} \left[ t^b V_0^{\text{unp}} t^a V_2^{\text{pol} \dagger} \right] U_1^{\text{unp} \, ba} \right\rangle (z') \right. \\
+ \left. \theta(x_{10} - x_{21}) \frac{1}{N_c} \left\langle \text{tr} \left[ V_0^{\text{unp}} V_2^{\text{unp} \dagger} \right] \text{tr} \left[ V_2^{\text{unp}} V_1^{\text{pol} \dagger} \right] \right\rangle (z') - N_c \left\langle \text{tr} \left[ V_0^{\text{unp}} V_1^{\text{pol} \dagger} \right] \right\rangle_{122} (z') \right\}
\]

\[ \langle \langle \ldots \rangle \rangle = \frac{1}{z s} \langle \ldots \rangle \]

\[ \rho'^2 = \frac{1}{z' s} \]
Polarized Dipole Evolution in the Large-\(N_c\) Limit

In the large-\(N_c\) limit the equations close, leading to a system of 2 equations:

\[
\begin{align*}
\frac{\partial}{\partial \ln z} G_{10}(z) &= 0 \\
\frac{\partial}{\partial \ln z'} \Gamma_{02,21}(z') &= 0
\end{align*}
\]

\[
G_{10}(z) = G_{10}^{(0)}(z) + \frac{\alpha_s N_c}{2\pi} \int_{z_i}^{z} \frac{d\z}{\z'} \int_{\rho^2}^{\rho'^2} \frac{dx_{10}^2}{x_{21}^2} [2 \Gamma_{02,21}(z') S_{21}(z') + 2 G_{21}(z') S_{02}(z') \\
&+ G_{12}(z') S_{02}(z') - \Gamma_{01,21}(z')]
\]

\[
\Gamma_{02,21}(z') = \Gamma_{02,21}^{(0)}(z') + \frac{\alpha_s N_c}{2\pi} \int_{z_i}^{z'} \frac{d\z''}{\z''} \int_{\rho'^2}^{\rho''^2} \frac{dx_{32}^2}{x_{32}^2} [2 \Gamma_{03,32}(z'') S_{23}(z'') + 2 G_{32}(z'') S_{03}(z'') \\
&+ G_{23}(z'') S_{03}(z'') - \Gamma_{02,32}(z'')]
\]

\(S = \) found from BK/JIMWLK, it is LLA
You friendly “neighborhood” dipole

• There is a new object in the evolution equation – the neighbor dipole.
• This is specific for the DLA evolution. Gluon emission may happen in one dipole, but, due to transverse distance ordering, may ‘know’ about another dipole:

\[ x_{21}^2 z' \gg x_{32}^2 z'' \]

• We denote the evolution in the neighbor dipole 02 by \( \Gamma_{02, 21}(z') \).
Large-$N_c$ Evolution

- In the strict DLA limit ($S=1$) and at large $N_c$ we get (here $\Gamma$ is an auxiliary function we call the ‘neighbour dipole amplitude’) (KPS ’15)

\[
G(x_{10}^2, z) = G^{(0)}(x_{10}^2, z) + \frac{\alpha_s N_c}{2\pi} \int \frac{dz'}{z'} \int \frac{dx_{21}^2}{x_{21}^2} \left[ \Gamma(x_{10}^2, x_{21}^2, z') + 3 G(x_{21}^2, z') \right]
\]

\[
\Gamma(x_{10}^2, x_{21}^2, z') = \Gamma^{(0)}(x_{10}^2, x_{21}^2, z') + \frac{\alpha_s N_c}{2\pi} \int \frac{dz''}{z''} \int \frac{dx_{32}^2}{x_{32}^2} \left[ \Gamma(x_{10}^2, x_{32}^2, z'') + 3 G(x_{32}^2, z'') \right]
\]

- The initial conditions are given by the Born-level graphs

\[
\Gamma^{(0)}(x_{10}^2, x_{21}^2, z) = G^{(0)}(x_{10}^2, z)
\]

\[
G^{(0)}(x_{10}^2, z) = \frac{\alpha_s^2 C_F}{N_c} \pi \left[ C_F \ln \frac{z s}{\Lambda^2} - 2 \ln(z s x_{10}^2) \right]
\]
Resummation Parameter

• For helicity evolution the resummation parameter is different from BFKL, BK or JIMWLK, which resum powers of leading logarithms (LLA)

\[ \alpha_s \ln(1/x) \]

• Helicity evolution resummation parameter is double-logarithmic (DLA):

\[ \alpha_s \ln^2 \frac{1}{x} \]

• The second logarithm of x arises due to transverse momentum (or transverse coordinate) integration being logarithmic both in the UV and IR.

• This was known before: Kirschner and Lipatov ’83; Kirschner ’84; Bartels, Ermolaev, Ryskin ‘95, ‘96; Griffiths and Ross ’99; Itakura et al ’03; Bartels and Lublinsky ‘03.
Quark Helicity at Small $x$

- These equations can be solved both numerically and analytically. (KPS ‘16–’17)

- The small-$x$ asymptotics of quark helicity is (at large $N_c$)

$$s_{10} = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{1}{x_{10}^2 \Lambda^2}$$

$$\eta = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{zs}{\Lambda^2}$$

- The small-$x$ asymptotics of quark helicity is (at large $N_c$)

$$\Delta q(x, Q^2) \sim \left( \frac{1}{x} \right)^{\alpha_h^q}$$

with

$$\alpha_h^q = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 2.31 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$
Intercepts

Here we plot our (flavor-singlet) helicity intercept as a function of the coupling. We show BER result and LO BFKL (all twist and leading twist) for comparison.
Small-x Evolution at large $N_c & N_f$

- At large $N_c & N_f$ there is no simple relation between the adjoint and fundamental polarized dipole amplitudes. We need to construct coupled evolution equations mixing them with each other.

- Here’s the adjoint dipole evolution:
Small-x Evolution at large $N_c$ & $N_f$

- At large $N_c$ & $N_f$ there is no simple relation between the adjoint and fundamental polarized dipole amplitudes. We need to construct coupled evolution equations mixing them with each other.

- Here’s the fundamental dipole evolution:
The resulting equations are

\[
Q_{10}(zs) = Q_{10}^{(0)}(zs) + \frac{\alpha_s N_c}{2\pi} \int \frac{dz'}{z'} \int \frac{dx_{21}^2}{x_{21}^2} \left\{ \frac{1}{2} \Gamma_{02,21}^{adj}(z') + \frac{1}{2} G_{21}^{adj}(z') + Q_{12}(z') - \bar{\Gamma}_{01,21}(z') \right\} + \frac{\alpha_s N_c}{4\pi} \int \frac{dz'}{z'} \int \frac{x_{10}^2 z'}{x_{21}^2} Q_{21}(z'),
\]

\[
G_{10}^{adj}(z) = G_{10}^{adj(0)}(z) + \frac{\alpha_s N_c}{2\pi} \max\{\Lambda^2, x_{10}^2\} \int \frac{dz'}{z'} \int \frac{x_{10}^2 z'}{x_{21}^2} \left[ \Gamma_{10,21}^{adj}(z') + 3 G_{21}^{adj}(z') \right] - \alpha_s N_f \frac{1}{2\pi} \Lambda^2 \int \frac{dz'}{z'} \int \frac{x_{10}^2 z'}{x_{21}^2} \bar{\Gamma}_{02,21}(z'),
\]

\[
\Gamma_{10,21}^{adj}(z') = \Gamma_{10,21}^{adj(0)}(z') + \frac{\alpha_s N_c}{2\pi} \max\{\Lambda^2, x_{10}^2\} \int \frac{dz''}{z''} \Gamma_{10,32}^{adj}(z'') \frac{dx_{32}^2}{x_{32}^2} \left[ \Gamma_{10,32}^{adj}(z'') + 3 G_{32}^{adj}(z'') \right] - \alpha_s N_f \frac{1}{2\pi} \Lambda^2 \int \frac{dz''}{z''} \Gamma_{03,32}^{adj}(z''),
\]

\[
\Gamma_{10,21}^{(0)}(z') = \Gamma_{10,21}^{(0)}(z') + \frac{\alpha_s N_c}{2\pi} \max\{\Lambda^2, x_{10}^2\} \int \frac{dz''}{z''} \min\{x_{10}^2, x_{21}^2, z'/z''\} \frac{dx_{32}^2}{x_{32}^2} \left\{ \frac{1}{2} \Gamma_{03,32}^{adj}(z'') + \frac{1}{2} G_{32}^{adj}(z'') + Q_{32}(z'') - \bar{\Gamma}_{01,32}(z) \right\} + \frac{\alpha_s N_c}{4\pi} \int \frac{dz''}{z''} \min\{x_{10}^2, x_{21}^2, z'/z''\} Q_{32}(z'').
\]

These are yet to be solved.
B. Gluon Helicity at Small $x$

Dipole Gluon Helicity TMD

• Now let us repeat the calculation for gluon helicity TMDs.

• We start with the definition of the gluon dipole helicity TMD:

\[
g_1^G(x, k_T^2) = \frac{-2i S_L}{x P^+} \int \frac{d\xi^- d^2\xi}{(2\pi)^3} e^{ixP^+\xi^- - ik \cdot \xi} \langle P, S_L | \epsilon_T^{ij} \text{tr} \left[ F^{+i}(0) U^{[+][0, \xi]} F^{-j}(\xi) U^{[-][\xi, 0]} \right] | P, S_L \rangle_{\xi^+=0}
\]

• Here \( U^{[+]}) \) and \( U^{[-]} \) are future and past Wilson line staples (hence the name `dipole’ TMD, F. Dominguez et al ’11 – looks like a dipole scattering on a proton):
Dipole Gluon Helicity TMD

• At small $x$, the definition of dipole gluon helicity TMD can be massaged into

$$ g_1^{G \, dip}(x, k_T^2) = \frac{8i N_c S_L}{g^2 (2\pi)^3} \int d^2 x_10 \, e^{i k \cdot x_10} \, k_\perp^i \epsilon_T^{ij} \left[ \int d^2 b_{10} \, G_{10}^j(zs = \frac{Q^2}{x}) \right] $$

• Here we obtain a new operator, which is a transverse vector (written here in $A^\perp=0$ gauge):

$$ G_{10}^i(z) \equiv \frac{1}{4N_c} \int_{-\infty}^{\infty} dx^- \left\langle \text{tr} \left[ V_0[\infty, -\infty] V_1[-\infty, x^-] (-ig) \tilde{A}^i(x^-, x) V_1[x^-, \infty] \right] + \text{c.c.} \right\rangle (z) $$

• Note that $k_\perp^i \epsilon_T^{ij}$ can be thought of as a transverse curl acting on $G_{10}^i(z)$ and not just on $\tilde{A}^i(x^-, x)$ -- different from the polarized dipole amplitude!
Dipole TMD vs dipole amplitude

• Note that the operator for the dipole gluon helicity TMD

\[ G_{10}^{i}(z) \equiv \frac{1}{4N_c} \int_{-\infty}^{\infty} dx^- \left\langle \text{tr} \left[ V_0[\infty, -\infty] V_1[-\infty, x^-] (-ig) \tilde{A}^i(x^-, x) V_1[x^-, \infty] \right] + \text{c.c.} \right\rangle (z) \]

is different from the polarized dipole amplitude

\[ G_{10}(z) \equiv \frac{1}{4N_c} \int_{-\infty}^{\infty} dx^- \left\langle \text{tr} \left[ V_0[\infty, -\infty] V_1[-\infty, x^-] (-ig) \nabla \times \tilde{A}(x^-, x) V_1[x^-, \infty] \right] + \text{c.c.} \right\rangle (z) \]

• We conclude that the dipole gluon helicity TMD does not depend on the polarized dipole amplitude! (Hence the ‘dipole’ name may not even be valid for such TMDs.)

• This is different from the unpolarized gluon TMD case.
Evolution Equation

• To construct evolution equation for the operator $G^i$ governing the gluon helicity TMD we resum similar (to the quark case) diagrams:
Large-$N_c$ Evolution: Diagrams

- At large-$N_c$ the equations are

$$\frac{\partial}{\partial Y} G_{10}(zs) = \Gamma_{20,21}^{\text{gen}}(z's) + \Gamma_{21,20}^{\text{gen}}(z's) + \Gamma_{10,21}(z's) + \text{“c.c.”}$$
Large-$N_c$ Evolution: Diagrams

- and

\[ \frac{\partial}{\partial Y} \Gamma_{10,21}(z') = \Gamma_{30,31}(z'') + \Gamma_{31,30}(z'') + \text{c.c.} \]

\[ \Gamma_{10,31}(z'') \]

\[ \Gamma_{30,31}(z'') \]

\[ \Gamma_{31,30}(z'') \]

\[ \text{c.c.} \]
Large-$N_c$ Evolution: Equations

- This results in the following evolution equations:

$$G_{10}^i(zs) = G_{10}^{i(0)}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int \frac{dz'}{z'} \int d^2x_2 \ln \frac{1}{x_{21}\Lambda} \frac{\epsilon_{ij}^T (x_{21})^j_1}{x_{21}^2} \left[ \Gamma_{20,21}^{\text{gen}}(z'zs) + G_{21}(z'zs) \right]$$

$$- \frac{\alpha_s N_c}{2\pi^2} \int \frac{dz'}{z'} \int d^2x_2 \ln \frac{1}{x_{21}\Lambda} \frac{\epsilon_{ij}^T (x_{20})^j_1}{x_{20}^2} \left[ \Gamma_{20,21}^{\text{gen}}(z'zs) + \Gamma_{21,20}^{\text{gen}}(z'zs) \right]$$

$$+ \frac{\alpha_s N_c}{2\pi} \int \frac{z}{z'} \int d^2x_2 \int \frac{dx_{21}^2}{x_{21}^2} \left[ G_{12}^i(z'zs) - \Gamma_{10,21}^i(z'zs) \right]$$

$$\Gamma_{10,21}^i(z'zs) = G_{10}^{i(0)}(z'zs) + \frac{\alpha_s N_c}{2\pi^2} \int \frac{dz''}{z''} \int d^2x_3 \ln \frac{1}{x_{31}\Lambda} \frac{\epsilon_{ij}^T (x_{31})^j_1}{x_{31}^2} \left[ \Gamma_{30,31}^{\text{gen}}(z''zs) + G_{31}(z''zs) \right]$$

$$- \frac{\alpha_s N_c}{2\pi^2} \int \frac{dz''}{z''} \int d^2x_3 \ln \frac{1}{x_{31}\Lambda} \frac{\epsilon_{ij}^T (x_{30})^j_1}{x_{30}^2} \left[ \Gamma_{30,31}^{\text{gen}}(z''zs) + \Gamma_{31,30}^{\text{gen}}(z''zs) \right]$$

$$+ \frac{\alpha_s N_c}{2\pi^2} \int \frac{z'}{z''} \int dx_{31}^2 \int \frac{dx_{31}^2}{x_{31}^2} \left[ G_{13}^i(z''zs) - \Gamma_{10,31}^i(z''zs) \right].$$
Large-\(N_C\) Evolution: Equations

- Here
  \[
  \Gamma_{20,21}^{\text{gen}}(z's) = \theta(x_{20} - x_{21}) \Gamma_{20,21}(z's) + \theta(x_{21} - x_{20}) G_{20}(z's)
  \]
is an object which we know from the quark helicity evolution, as the latter gives us \(G\) and \(\Gamma\).

- Note that our evolution equations mix the gluon \((G^i)\) and quark \((G)\) small-x helicity evolution operators:
  \[
  G^i_{10}(z's) = G^i_{10}(0)(z's) + \frac{\alpha_s N_c}{2\pi^2} \int_{\frac{\Delta^2}{s}}^z \frac{dz'}{z'} \int d^2 x_2 \ln \frac{1}{x_{21}\Lambda} \frac{\epsilon^{ij}_{T}(x_{21})^j}{x_{21}^2} \left[ \Gamma_{20,21}^{\text{gen}}(z's) + G_{21}(z's) \right]
  \]
  \[
  - \frac{\alpha_s N_c}{2\pi^2} \int_{\frac{\Delta^2}{s}}^z \frac{dz'}{z'} \int d^2 x_2 \ln \frac{1}{x_{21}\Lambda} \frac{\epsilon^{ij}_{T}(x_{20})^j}{x_{20}^2} \left[ \Gamma_{20,21}^{\text{gen}}(z's) + \Gamma_{21,20}^{\text{gen}}(z's) \right]
  \]
  \[
  + \frac{\alpha_s N_c}{2\pi} \int_{\frac{1}{x_{10}^2 s}}^z \frac{dz'}{z'} \int \frac{dx_{21}^2}{x_{21}^2} \left[ G^i_{12}(z's) - \Gamma_{10,21}^i(z's) \right]
  \]
Initial Conditions

- Initial conditions for this evolution are given by the lowest order $t$-channel gluon exchanges:

\[
\int d^2b_{10} \, G_{10}^{(0)}(zs) = \int d^2b_{10} \, \Gamma_{10,21}^{(0)}(zs) = -\frac{\alpha_s^2 e_F}{N_c} \pi \epsilon^{ij} x_{10}^i x_{10}^j \ln \frac{1}{x_{10}^i}\]

- Note that these initial conditions have no $\ln s$, unlike the initial conditions for the quark evolution:

\[
\int d^2b_{10} \, G_{10}^{(0)}(zs) = \int d^2b_{10} \, \Gamma_{10,21}^{(0)}(zs) = -\frac{\alpha_s^2 e_F}{N_c} \pi \ln(zs x_{10}^2)\]
Large-$N_c$ Evolution: Power Counting

- The kernel mixing $G^i$ or $\Gamma^i$ with $G$ and $\Gamma$ is LLA:

$$
G^{i\to 10}_i(z s) = G^{i\to (0)}_i(z s) + \frac{\alpha_s N_c}{2\pi^2} \int z' \int \frac{dz'}{z'} \int d^2 x_2 \ln \frac{1}{x_{21}\Lambda} \frac{\epsilon^{ij}_T (x_{21})^j_{\perp}}{x_{21}^2} \left[ \Gamma^{gen}_{20, 21}(z' s) + G_{21}(z' s) \right] 
$$

$$
- \frac{\alpha_s N_c}{2\pi^2} \int \left[ \frac{dz}{z} \int d^2 x_2 \ln \frac{1}{x_{21}\Lambda} \frac{\epsilon^{ij}_T (x_{20})^j_{\perp}}{x_{20}^2} \left[ \Gamma^{gen}_{20, 21}(z' s) + \Gamma^{gen}_{21, 20}(z' s) \right] 

+ \frac{\alpha_s N_c}{2\pi} \int \frac{dz}{z} \int \frac{d^2 x_2}{x_{21}^2} \left[ G^{i\to z_0}_i(z' s) - \Gamma^{i\to 10, 21}_i(z' s) \right] 
$$

- But, the initial conditions for $G$ and $\Gamma$ have an extra $\ln s$ as compared to $G^i$ and $\Gamma^i$, making the two terms comparable (order-$\alpha_s^2$ in $\alpha_s \ln^2 s \sim 1$ DLA power counting).
Large-\(N_c\) Evolution Equations: Solution

- These equations can be solved in the asymptotic high-energy region yielding the small-\(x\) gluon helicity intercept

\[
\alpha_h^G = \frac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 1.88 \sqrt{\frac{\alpha_s N_c}{2\pi}}
\]

- We obtain the small-\(x\) asymptotics of the gluon helicity distributions:

\[
\Delta G(x, Q^2) \sim g_{1L}^{G dip}(x, k_T^2) \sim \left(\frac{1}{x}\right)^{\frac{13}{4\sqrt{3}}} \sqrt{\frac{\alpha_s N_c}{2\pi}}
\]
Main Physics Results

• At large $N_c$ we get for helicity

$$
\Delta q(x, Q^2) \sim \left( \frac{1}{x} \right)^{\alpha_h^q} \quad \text{with} \quad \alpha_h^q = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 2.31 \sqrt{\frac{\alpha_s N_c}{2\pi}}
$$

$$
\Delta G(x, Q^2) \sim \left( \frac{1}{x} \right)^{\alpha_h^G} \quad \text{with} \quad \alpha_h^G = \frac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 1.88 \sqrt{\frac{\alpha_s N_c}{2\pi}}
$$

• For valence quark transversity TMDs we have (also at large $N_c$)

$$
h_{1T}^{NS}(x, k_T^2) \sim h_{1T}^{\perp NS}(x, k_T^2) \sim \left( \frac{1}{x} \right)^{\alpha_t^q} \quad \text{with} \quad \alpha_t^q = -1 + 2 \sqrt{\frac{\alpha_s C_F}{\pi}}
$$
Orbital Angular Momentum

• The small-$x$ asymptotics of the quark and gluon orbital angular momentum (OAM) in the Jaffe-Manohar decomposition shown above can be tackled similarly (YK, 2019).

• In the large-$N_c$ limit we obtain

\[
L_{q+\bar{q}}(x, Q^2) = -\Delta \Sigma(x, Q^2) \sim \left( \frac{1}{x} \right) \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}},
\]

\[
L_G(x, Q^2) \sim \Delta G(x, Q^2) \sim \left( \frac{1}{x} \right)^{\frac{13}{4\sqrt{3}}} \sqrt{\frac{\alpha_s N_c}{2\pi}}.
\]
C. Some Spin Phenomenology at Small \( x \)

Impact of our $\Delta \Sigma$ on the proton spin

- We have attached a $\Delta \tilde{\Sigma}(x, Q^2) = N x^{-\alpha_h}$ curve to the existing hPDF’s fits at some ad hoc small value of $x$ labeled $x_0$:
Impact of our $\Delta \Sigma$ on the proton spin

- Defining $\Delta \Sigma^{[x_{min}]}(Q^2) \equiv \int_{x_{min}}^{1} dx \, \Delta \Sigma(x, Q^2)$ we plot it for $x_0=0.03$, 0.01, 0.001:

- We observe a moderate to significant enhancement of quark spin.
- More detailed phenomenology is needed in the future.
Here we compare our results for the total quark helicity with DSSV, now including their error band.

We observe consistency of our lower two curves with DSSV.

Our upper curve disagrees with DSSV, but agrees with NNPDF (Nocera, Santopinto, ‘16).

Better phenomenology is needed. EIC would definitely play a role.
Impact of our $\Delta G$ on the proton spin

- We have attached a $\Delta G(x, Q^2) = N x^{-\alpha_h^G}$ curve to the existing hPDF's fits at some ad hoc small value of $x$ labeled $x_0$: 

\[ x\Delta g(x, Q^2) \]

\begin{figure}
\centering
\includegraphics[width=\textwidth]{impact_of_dg_on_proton_spin}
\caption{Impact of our $\Delta G$ on the proton spin}
\end{figure}

“ballpark” phenomenology
Impact of our $\Delta G$ on the proton spin

- Defining $S_G^{[x_{min}]}(Q^2) \equiv \int_{x_{min}}^{1} dx \Delta G(x, Q^2)$ we plot it for $x_0=0.08, 0.05, 0.001$:

- We observe a moderate enhancement of gluon spin.
- More detailed phenomenology is needed in the future.
• Parton helicity distributions are sensitive to low-x physics.
• EIC would have an unprecedented low-x reach for a polarized DIS experiment, allowing to pinpoint the values of quark and gluon contributions to proton’s spin:

\[ \Delta G \text{ and } \Delta \Sigma \text{ are integrated over } x \text{ in the } 0.001 < x < 1 \text{ interval.} \]
EIC: Solving the Spin Puzzle

1/2 - Gluon - Quarks = orbital angular momentum

- Above plot shows the running integral of $\Delta g(x,Q^2)$ from $x_{\text{min}}$ to 1 as a function of $x_{\text{min}}$
- Large reduction in uncertainty on $\Delta G$ from EIC can be seen
- EIC will also reduce the uncertainty on the quark contribution to the proton spin

Constraints on gluon and quark contributions will provide information on the orbital angular momentum component of proton spin

Conclusions

• The small-x field has evolved tremendously over the last three decades, with the community making real conceptual progress in understanding QCD in high energy hadronic and nuclear collisions.

• High energy collisions probe a dense system of gluons (Color Glass Condensate), described by nonlinear BK/JIMWLK evolution equations with highly non-trivial behavior.

• Progress in understanding higher order corrections led to an amazingly good agreement of saturation physics fits and predictions (!) with many DIS, p+A, and A+A experiments at HERA, RHIC, and LHC.

• The field has been expanding into other sub-fields, including spin physics, where the small-x behavior of many important observables can now be tackled. So far the leading small-x asymptotics have been found for quark and gluon helicity and OAM, and for valence quark transversity, in addition to a number of unpolarized quark and gluon TMDs and linearly polarized gluon TMDs.
Backup Slides
More Recent Progress
A. Running Coupling
Non-linear evolution: fixed coupling

- Theoretically nothing is wrong with it: preserves unitarity (black disk limit), prevents the IR catastrophe.
- Phenomenologically there is a problem though: LO BFKL intercept is way too large (compared to 0.2-0.3 needed to describe experiment)

\[ \alpha_P - 1 = 2.77 \frac{\alpha_s N_c}{\pi} \approx 0.79 \]

- Full NLO calculation (order-\(\alpha^2\) kernel): tough, but done (see Balitsky and Chirilli ’07).

- First let’s try to determine the scale of the coupling.
What Sets the Scale for the Running Coupling?

\[
\frac{\partial N(x_0, x_1, Y)}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2} \int d^2 x_2 \frac{x_{01}^2}{x_{02}^2 x_{12}^2} \\
\times [N(x_0, x_2, Y) + N(x_2, x_1, Y) - N(x_0, x_1, Y) - N(x_0, x_2, Y) N(x_2, x_1, Y)]
\]

transverse plane
What Sets the Scale for the Running Coupling?

\[ \frac{\partial N(x_0, x_1, Y)}{\partial Y} = \frac{\alpha_s N_C}{2 \pi^2} \int d^2 x_2 \frac{x_{01}^2}{x_{02} x_{12}} \times [N(x_0, x_2, Y) + N(x_2, x_1, Y) - N(x_0, x_1, Y) - N(x_0, x_2, Y) N(x_2, x_1, Y)] \]

**\(\alpha_s(???)\)**

In order to perform consistent calculations, it is important to know the scale of the running coupling constant in the evolution equation.

There are three possible scales – the sizes of the “parent” dipole and “daughter” dipoles \(x_{01}, x_{21}, x_{20}\). Which one is it?
• The answer is that the running coupling corrections come in as a “triumvirate” of couplings (H. Weigert, Yu. K. ’06; I. Balitsky, ‘06):

\[ \alpha_\mu \Rightarrow \frac{\alpha_s(\ldots) \alpha_s(\ldots)}{\alpha_s(\ldots)} \]

cf. Braun ’94, Levin ‘94

• The scales of three couplings are somewhat involved.
Main Principle

To set the scale of the coupling constant we will first calculate the $\alpha_s N_f$ corrections to BK/JIMWLK evolution kernel to all orders.

We then would complete $N_f$ to the QCD beta-function

$$\beta_2 = \frac{11 N_C - 2 N_f}{12 \pi}$$

by replacing $N_f \rightarrow -6 \pi \beta_2$ to obtain the scale of the running coupling:

$$\alpha_s(Q^2) = \frac{\alpha_\mu}{1 + \alpha_\mu \beta_2 \ln(Q^2/\mu^2)}$$

BLM prescription (Brodsky, Lepage, Mackenzie ’83)
Running Coupling Corrections to All Orders

One has to insert fermion bubbles to all orders:

\[ U_{ax} t^a \otimes t^a U_{yx}^\dagger \]

\[ U_{bx} t^a \otimes t^a U_{yx}^\dagger \]

\[ 2 \text{tr} (t^a U_{z1} t^b U_{z2}^\dagger) \times t^a U_{ax} \otimes t^b U_{yx}^\dagger \]
The resulting JIMWLK kernel with running coupling corrections is

\[ \alpha_\mu K(x_0, x_1; z) = 4 \int \frac{d^2 q d^2 q'}{(2\pi)^4} e^{-iq(z-x_0)+iq'(z-x_1)} \frac{q \cdot q'}{q^2 q'^2} \frac{\alpha_S(q^2) \alpha_S(q'^2)}{\alpha_S(Q^2)} \]

where

\[ \ln \frac{Q^2}{\mu^2} = \frac{q^2 \ln(q^2 / \mu^2) - q'^2 \ln(q'^2 / \mu^2)}{q^2 - q'^2} - \frac{q^2 q'^2}{q \cdot q'} \frac{\ln(q^2 / q'^2)}{q^2 - q'^2} \]

The BK kernel is obtained from the above by summing over all possible emissions of the gluon off the quark and anti-quark lines.

The resulting JIMWLK kernel with running coupling corrections is

\[ \alpha_\mu K(x_0, x_1; z) = 4 \int \frac{d^2 q d^2 q'}{(2\pi)^4} e^{-iq(z-x_0)+iq'(z-x_1)} \frac{q \cdot q'}{q^2 q'^2} \frac{\alpha_S(q^2) \alpha_S(q'^2)}{\alpha_S(Q^2)} \]

where

\[ \ln \frac{Q^2}{\mu^2} = \frac{q^2 \ln(q^2 / \mu^2) - q'^2 \ln(q'^2 / \mu^2)}{q^2 - q'^2} - \frac{q^2 q'^2}{q \cdot q'} \frac{\ln(q^2 / q'^2)}{q^2 - q'^2} \]

The BK kernel is obtained from the above by summing over all possible emissions of the gluon off the quark and anti-quark lines.
Running Coupling BK

Here’s the BK equation with the running coupling corrections (H. Weigert, Yu. K. ’06; I. Balitsky, ‘06):

\[
\frac{\partial N(x_0, x_1, Y)}{\partial Y} = \frac{N_C}{2\pi^2} \int d^2 x_2
\times \left[ \frac{\alpha_s(1/x_{02}^2)}{x_{02}^2} + \frac{\alpha_s(1/x_{12}^2)}{x_{12}^2} - 2 \frac{\alpha_s(1/x_{02}^2) \alpha_s(1/x_{12}^2)}{\alpha_s(1/R^2)} \frac{x_{20} \cdot x_{21}}{x_{02} x_{12}} \right]
\times [N(x_0, x_2, Y) + N(x_2, x_1, Y) - N(x_0, x_1, Y) - N(x_0, x_2, Y) N(x_2, x_1, Y)]
\]

where

\[
\ln R^2 \mu^2 = \frac{x_{20}^2 \ln(x_{21}^2 \mu^2) - x_{21}^2 \ln(x_{20}^2 \mu^2)}{x_{20}^2 - x_{21}^2} + \frac{x_{20}^2 x_{21}^2}{x_{20} \cdot x_{21}} \frac{\ln(x_{20}^2 / x_{21}^2)}{x_{20}^2 - x_{21}^2}
\]
What does the running coupling do?

- Slows down the evolution with energy / rapidity.

\[ \lambda = \frac{d \ln Q_s^2(Y)}{dY} \]

down from about

\[ \lambda \approx 0.7 \div 0.8 \]

at fixed coupling

Albacete ‘07
Solution of the Full Equation

Different curves – different ways of separating running coupling from NLO corrections. Solid curve includes all corrections.

J. Albacete, Yu.K. ‘07
At high enough rapidity we recover geometric scaling, all solutions fall on the same curve. This has been known for fixed coupling: however, the shape of the scaling function is different in the running coupling case!

J. Albacete, Yu.K. ‘07
B. NLO BFKL/BK/JIMWLK
NLO BK

• NLO BK evolution was calculated by Balitsky and Chirilli in 2007.
• It resums powers of $\alpha_s^2 Y$ (NLO) in addition to powers of $\alpha_s Y$ (LO).
• Here’s a sampler of relevant diagrams (need kernel to order-$\alpha^2$):

Diagrams with 2 gluons interaction
NLO BK

• The large-$N_C$ limit:

\[
\frac{d}{d\eta} N(x, y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2z \frac{(x-y)^2}{X^2 Y^2} \left\{ 1 + \frac{\alpha_s N_c}{4\pi} \left[ \frac{11}{3} \ln(x-y)^2 \mu^2 - \frac{11}{3} \frac{X^2 - Y^2}{(x-y)^2} \ln Y^2 + \frac{67}{9} - \frac{\pi^2}{3} - 2 \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \right] \right\} \\
\times \left[ N(x, z) + N(z, y) - N(x, y) - N(x, z)N(z, y) \right] \\
+ \frac{\alpha_s^2 N_c^2}{8\pi^4} \int d^2z d^2z' \left\{ -\frac{2}{(z-z')^4} + \frac{X^2 Y^2 + X'^2 Y^2 - 4(x-y)^2(z-z')^2}{(z-z')^4(X^2 Y^2 - X'^2 Y^2)} + \frac{(x-y)^4}{X^2 Y^2(X^2 Y^2 - X'^2 Y^2)} \right\} \\
+ \frac{(x-y)^2}{X^2 Y^2(z-z')^2} \ln \frac{X^2 Y^2}{X'^2 Y^2} \left[ N(z, z') - N(x, z)N(z, z') - N(z, z')N(z', y) - N(x, z)N(z', y) + N(x, z)N(z, y) \right] \\
+ N(x, z)N(z, z')N(z', y). \right\} 
\]

(yet to be solved numerically)
NLO JIMWLK

• NLO evolution has been calculated for other Wilson line operators (not just dipoles), most notably the 3-Wilson line operator (Grabovsky ‘13, Balitsky & Chirilli ’13, Kovner, Lublinsky, Mulian ’13, Balitsky and Grabovsky ‘14).

• The NLO JIMWLK Hamiltonian was constructed as well (Kovner, Lublinsky, Mulian ’13, ’14).

• However, the equations do not close, that is, the operators on the right hand side can not be expressed in terms of the operator on the left. Hence can’t solve.

• To find the expectation values of the corresponding operators, one has to perform a lattice calculation with the NLO JIMWLK Hamiltonian, generating field configurations to be used for averaging the operators.
NLO Dipole Evolution at any $N_C$

- NLO BK equation is the large-$N_C$ limit of (Balitsky and Chrilli ’07)
Summary

• Running coupling and NLO corrections have been calculated for BK and JIMWLK equations.

• \( rc_{BK} \) and \( rc_{JIMWLK} \) have been solved numerically and used in phenomenology (DIS, pA, AA) with reasonable success.

• NLO BK and NLO JIMWLK have not yet been solved.
Conclusions

• In these lectures I introduced a 3-step approach to CGC: classical physics, small-x evolution, and running coupling corrections.

• This prescription appears to describe a wide range of small-x data on DIS, p(d)A, and AA collisions.
Last time

• The equation reads:

\[ \partial_Y N_{x_0, x_1}(Y) = \frac{\alpha_s N_c}{2 \pi^2} \int d^2 x_2 \frac{x_{01}^2}{x_{02} x_{21}^2} \left[ N_{x_0, x_2}(Y) + N_{x_2, x_1}(Y) - N_{x_0, x_1}(Y) - N_{x_0, x_2}(Y) N_{x_2, x_1}(Y) \right] \]

• It combines BFKL evolution (the linear part) and the quadratic damping correction.

• We discussed its solution: