Update on $a_\mu^{\mathrm{HVP,isoQCD}}$ from the ETM collaboration

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July-August 2026

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Introduction

$g_\mu - 2$: benchmark of precision physics
Aliberti et al., 2025:

$$a_\mu^{\text{SM}} = 116592033(62) \cdot 10^{-11}$$
$$a_\mu^{\text{exp}} = 116592071.5(14.5) \cdot 10^{-11}$$

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Main source of uncertainty coming from:

$$a_\mu^{\text{had}} = a_\mu^{\text{HVP}} + a_\mu^{\text{HLbL}}$$

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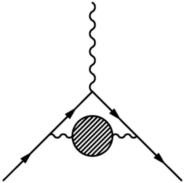
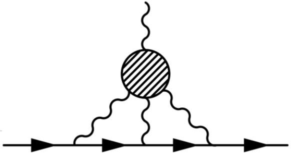


Hadronic Light by Light (HLbL)



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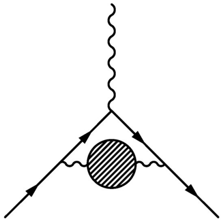

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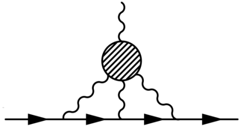
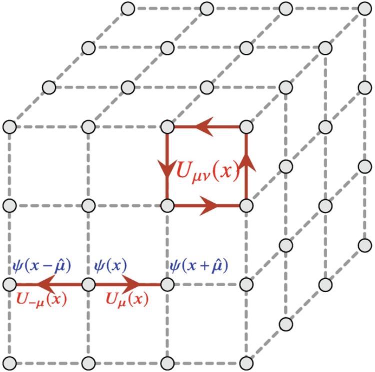


Hadronic contribution from the lattice



$$S[U,\psi,\bar{\psi}] = S_{\mathrm{YM}}[U] + S_F[\psi,\bar{\psi},U]$$

- a_μ^{had} is led by non-perturbative effects.
- A lattice calculation provides a first-principles tool to perform the calculation
- Physical predictions are found by extrapolating to the continuum ($a \to 0$) and infinite volume ($V \to \infty$):

$$\langle O \rangle = \frac{\int \mathcal{D}U\, O[U,\psi,\bar{\psi}]\, e^{-S[U,\psi,\bar{\psi}]}}{\int \mathcal{D}U e^{-S[U,\psi,\bar{\psi}]}}$$

a_μ^{HVP} from the lattice

$$a_\mu^{\mathrm{HVP}} = 2\alpha_{em}^2 \int_0^\infty dt\, t^2\, K(m_\mu t)\, V(t)$$

$$K(z) = 2\int_0^1 dy(1-y)\left[1 - j_0^2\left(\frac{z}{2}\frac{y}{\sqrt{1-y}}\right)\right] \,, \qquad j_0(\xi) = \frac{\sin(\xi)}{\xi}$$

From the lattice

$$V(t) \equiv \frac{1}{3}\sum_{i=1,2,3}\int d^3x\, \langle J_i(\vec{x},t) J_i^\dagger(0)\rangle \qquad \text{(Wick contractions)}$$

with $J_i(x)$ being the electromagnetic current operator

$$J_i(x) \equiv \sum_{f=u,d,s,c,\dots} q_{\mathrm{em},f}\, j_i^f(x) \qquad j_i^f(x) = \overline{\psi}_f(x)\gamma_i\psi_f(x) \qquad i = 1,2,3$$

isoQCD + QED

- **Theory**: The action S should include all the SM

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- **Practice**: Gravitational and EW effects in the HVP are subleading. QED is a small perturbation on top of the **isoQCD**: QCD (only) with $m_\ell = m_u = m_d$.

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Including Isosping Breaking effects

When isoQCD precision reaches $O(1\%)$ we have to account for $m_u \neq m_d$ and $\alpha_{\mathrm{EM}} \neq 0$.

- *Leading Isospin Breaking Effects* (LIBEs): combination of 2 $O(1\%)$ effects
- isoQCD precision has to be at least of per-mille level.

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ETMC ensembles are at the isoQCD physical point up to $O(1\%)$ effects.
$\Longrightarrow$ We need to correct the mistuning of the parameters in the action

Definition of the isoQCD physical point

This talk: isoQCD contribution to a_μ^{HVP}.

Edinburgh/FLAG consensus

The scale is set through the pion decay constant ["Converging on QCD+QED prescriptions" 2023; Y. Aoki et al., 2024]:

$$\boxed{F_\pi^{\mathrm{iso}} = 130.5\ \mathrm{MeV}\,, \qquad a = \frac{(aF_\pi)}{F_\pi^{\mathrm{iso}}}}$$

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WP25 scheme

The scale is set through w_0 [Aliberti et al., 2025]:

$$\boxed{w_0^{\mathrm{iso}} = 0.17236\,\mathrm{fm}\,, \qquad a = \frac{w_0^{\mathrm{iso}}}{(w_0/a)}}$$

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In both schemes:

$$M_\pi^{\mathrm{iso}} = 135.0 \text{ MeV}, \quad M_K^{\mathrm{iso}} = 494.6 \text{ MeV}, \quad M_{D_s}^{\mathrm{iso}} = 1967 \text{ MeV}.$$

Fermion setup

ETMC uses **twisted-mass** (TM) fermions for sea quarks:

$$D_\ell^{\mathrm{TM}} = \gamma_\mu \nabla_\mu - i\gamma_5\tau^3\big(W_{\mathrm{cl}} + m_0\big) + \mu_\ell$$

- Strictly positive determinant for $\mu \neq 0$
- $m_0 = m_{\mathrm{cr}} \to$ **maximal twist** $\to$ automatic $O(a)$ improvement

Two discretizations for $V(t)$

- **TM**: valence TM fermions, renormalization with Z_A
- **OS** (Osterwalder–Seiler): valence OS fermions, renormalization with Z_V

$V_{\mathrm{TM}}(t)$ and $V_{\mathrm{OS}}(t)$ differ by $O(a^2)$, continuum limit taken simultaneously in a correlated fit

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Isospin decomposition: isoscalar vs isovector

Flavor decomposition reorganized in definite isospin:

$$J_\mu^{I=1}(x) = \frac{1}{2}\left[j_\mu^u(x) - j_\mu^d(x)\right] \qquad \text{(isovector)}$$

The EM current separates as $J_\mu^{\rm em} = J_\mu^{I=1} + J_\mu^{I=0}$, where the isoscalar $J^{I=0}$ contains the remaining flavor-diagonal contributions.

$$\boxed{a_\mu^{\rm HVP}(I{=}1) = \frac{9}{10}\, a_\mu^{\rm HVP}(\ell)\,, \qquad a_\mu^{\rm HVP}(I{=}0) = \frac{1}{10}\, a_\mu^{\rm HVP}(\ell) + a_\mu^{\rm HVP}(s+c+\text{disc.})}$$

Why separate? [Kuberski, 2024]

- Two-pion states → only $I{=}1$: sizeable FV effects and M_π-dependence cancel in $I{=}0$
- $V^{I=1}(t)$, $V^{I=0}(t)$ separately positive definite → bounding method applicable per channel

The $I = 0$ channel

Convenient to reorganize using the octet current [Djukanovic et al., 2025]

$$J^8_\mu(x) = \frac{1}{\sqrt{3}}\left[\frac{1}{2}\big(j^u_\mu(x) + j^d_\mu(x)\big) - j^s_\mu(x)\right]$$

$$\boxed{a^{\mathrm{HVP}}_\mu(I{=}0) = a^{\mathrm{HVP}}_\mu(\mathrm{oct.}) + a^{\mathrm{HVP}}_\mu(c) + \delta a^{c,\mathrm{disc}}_\mu}$$

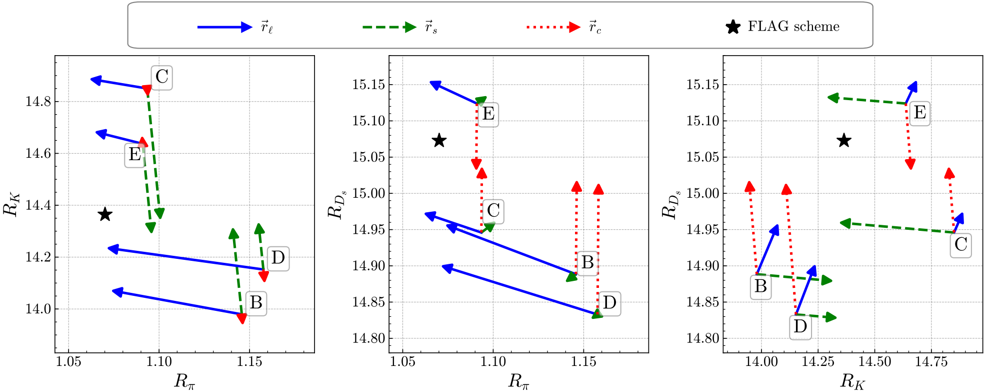
- $a^{\mathrm{HVP}}_\mu(\mathrm{oct.})$: from J^8_μ, disconnected light + strange loops, OS only
- $a^{\mathrm{HVP}}_\mu(c)$: charm-connected, TM and OS (as for $I = 1$)
- $\delta a^{c,\mathrm{disc}}_\mu$: residual disconnected charm, short-distance dominated $\rightarrow$ negligible

Strange and charm connected contributions already presented in [Alexandrou et al., 2024]

$$a^{\mathrm{HVP}}_\mu(s) = 53.57(63) \times 10^{-10}\,, \qquad a^{\mathrm{HVP}}_\mu(c) = 14.56(13) \times 10^{-10}$$

Tuning to the isoQCD physical point

$$\vec{R}^{\text{sim}} + \sum_{f=\ell,s,c} (m_f - m_f^{\text{sim}}) \frac{d\vec{R}}{dm_f}\bigg|_{\text{sim}} + (m_0 - m_0^{\text{sim}}) \frac{d\vec{R}}{dm_0}\bigg|_{\text{sim}} = \vec{R}^{\text{FLAG}}$$



Windows

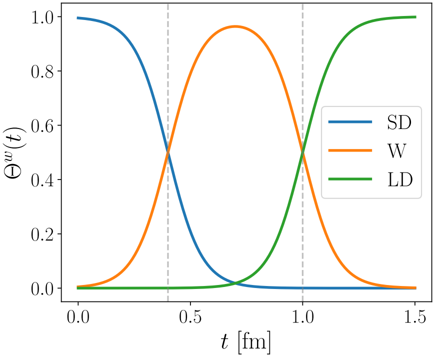
$$a_\mu^{\mathrm{HVP},w} = 2\alpha_{\mathrm{em}}^2 \int_0^\infty dt\, t^2 K(m_\mu t)\, \Theta^w(t)\, V(t)$$

$$\Theta^{\mathrm{SD}}(t) = 1 - \frac{1}{1+e^{-2(t-t_0)/\Delta}}$$

$$\Theta^{\mathrm{W}}(t) = \frac{1}{1+e^{-2(t-t_0)/\Delta}} - \frac{1}{1+e^{-2(t-t_1)/\Delta}}$$

$$\Theta^{\mathrm{LD}}(t) = \frac{1}{1+e^{-2(t-t_1)/\Delta}}$$

$$t_0 = 0.4 \text{ fm}, \quad t_1 = 1.0 \text{ fm}, \quad \Delta = 0.15 \text{ fm}$$



Reference-volume interpolation

Two volumes at coarsest spacing: **B64** ($L \simeq 5.1$ fm), **B96** ($L \simeq 7.6$ fm)

- Constrain FV effects at fixed a
- Interpolate B64 $\to$ common $L_{\text{ref}} \simeq 5.46$ fm
- Continuum extrapolation at fixed L_{ref}

Interpolation ansatz:

$$a_\mu^{\text{HVP}}(I{=}1; L) = a_\mu^{\text{HVP}}(I{=}1; \infty) + A L e^{-M_\pi L}$$

with A determined from the B64–B96 difference at fixed a

$I = 1$: Two-pion states $\to$ sizeable FV effects

$I = 0$: Octet channel far less L-sensitive

- **UV corrections**: subtract residual short-distance contributions

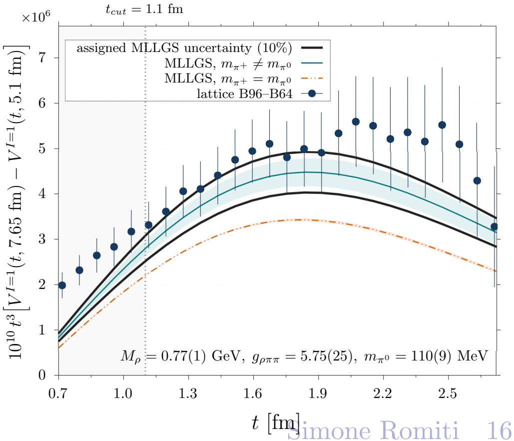
Residual FV correction: $L_{\mathrm{ref}} = 5.46$ fm $\to \infty$

MLLGS model [Harvey B. Meyer, 2011; Lellouch and Luscher, 2001; Luscher, 1991; Gounaris and Sakurai, 1968]

- For TM fermions, dominated by the lowest-lying $\pi^+\pi^0$ **energy states**

$$\Delta a_\mu^{\mathrm{HVP}}(L) = a_\mu^{\mathrm{HVP}}(\infty) - a_\mu^{\mathrm{HVP}}(L)$$

$$\omega(k) = \sqrt{m_{\pi^+}^2 + k^2} + \sqrt{m_{\pi^0}^2 + k^2}$$



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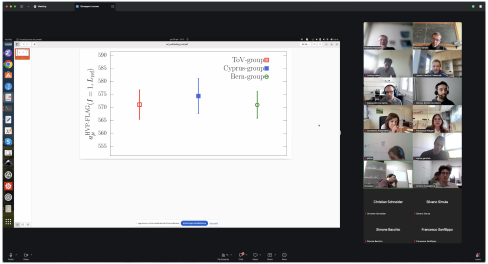
Blinding procedure

Exponential signals added to $V_\ell(t)$ **before** analysis:

$$V_\ell(t) \to V_\ell(t) + \frac{1}{Z}\sum_i A_i e^{-m_i t}$$

Shift identical on all lattice spacings.

Unblinding



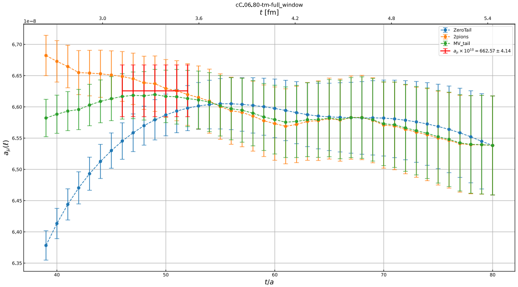
Blinding

- 3 independent groups: **Bern, Cyprus, Roma Tre**
- $\{A_i, m_i\}$ unknown to each group
- Cross-checked before unblinding

Bounding method [Bernecker and H. B. Meyer, 2011]

$$V^X(t_c)\, e^{-m_{\text{eff}}(t_*)(t-t_c)} \;\leq\; V^X(t) \;\leq\; V^X(t_c)\, e^{-E_0^X\,(t-t_c)}$$

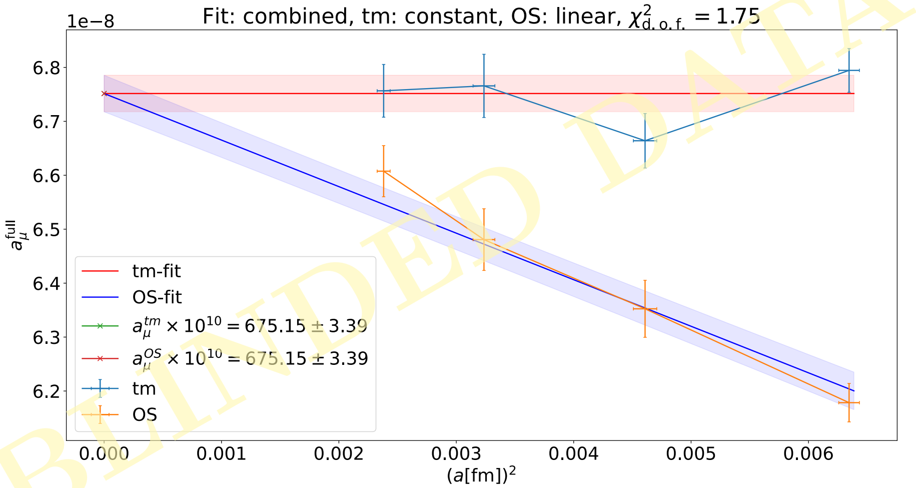
- **Upper bound:** 2-free-pions state on the lattice
- **Lower bound:** Use m_{eff} for $t > t_{\text{thr}}$



Strategies

- $t_{\text{thr}} = 1.8, 2.0$ fm
- Aggressive ($< 0\sigma$), Moderate ($< 1\sigma$)
- Plateau $0.25, 0.5$ fm

Continuum limit



Error budget via marginalization

- Each continuum-limit model carries an AIC weight w_m
- For each systematic k, **marginalize** over model choices

Law of total variance

$$\sigma^2_{\text{tot}}\Big|_k = \underbrace{\mathbb{E}[\mathrm{Var}(a_\mu|k)]}_{\text{stat. + other syst.}} + \underbrace{\mathrm{Var}(\mathbb{E}[a_\mu|k])}_{\text{syst. from } k} = \sum_m w_m \sigma_m^2 + \sum_m w_m (a_{\mu,m} - \bar{a}_{\mu,k})^2$$

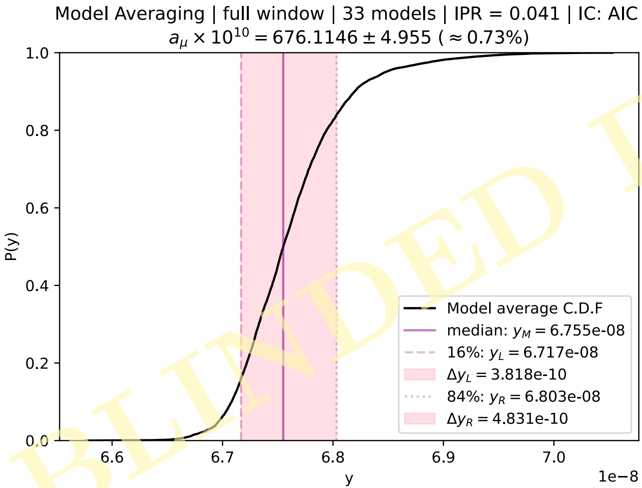
- **Estimator of the mean** (AIC-weighted):

$$\mathbb{E}[a_\mu|k] = \sum_{m\in\mathcal{M}_k} w_m a_{\mu,m}$$

Caveat

Individual systematics need not sum in quadrature; correlations between k (e.g. t_{thr} and bounding strategy) are monitored and found to be small.

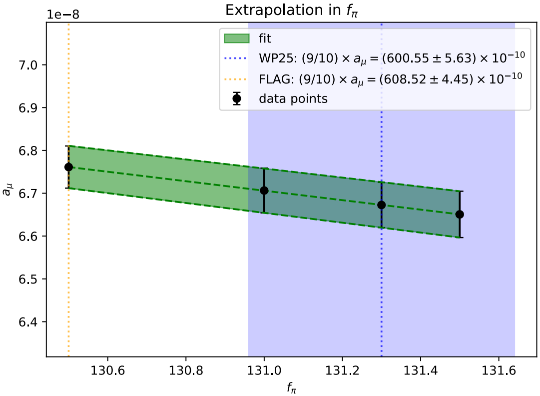
Error budget table



Error budget	$\Delta a_\mu \cdot 10^{10}$
tot	4.9550
stat	4.6139
syst_tot	1.8067
n_ens	0.0000
t_thr	0.1052
t_cut	0.1317
dt_plateau	0.1207
extrapolation	1.7499

Interpolation to WP25 scheme

- Analysis at $f_\pi = \{130.5, 131.0, 131.3, 131.5\}$ MeV
- Linear interpolation to $f_\pi^{\mathrm{WP25}} = 131.1(2)$ MeV



a_μ slope

$$\frac{\partial a_\mu^{\mathrm{HVP}}(I{=}1)}{\partial f_\pi} = -1.01(15) \times 10^{-10}\,\mathrm{MeV}^{-1}$$

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Summary of results

Individual $I = 1$ results:

$$a_\mu^{\mathrm{HVP,FLAG}}(I{=}1) = 597.4(6.0) \times 10^{-10}\,, \qquad a_\mu^{\mathrm{HVP,WP25}}(I{=}1) = 591.5(6.6) \times 10^{-10}$$

- Consistent with BMW20 and other lattice determinations within uncertainties
- Adding the WP25 leading isospin-breaking correction:

$$a_\mu^{\mathrm{HVP}} = a_\mu^{\mathrm{HVP}}(\mathrm{iso}) + a_\mu^{\mathrm{HVP}}(\mathrm{IB,\ WP25}) = (XX \pm YY) \times 10^{-10}$$

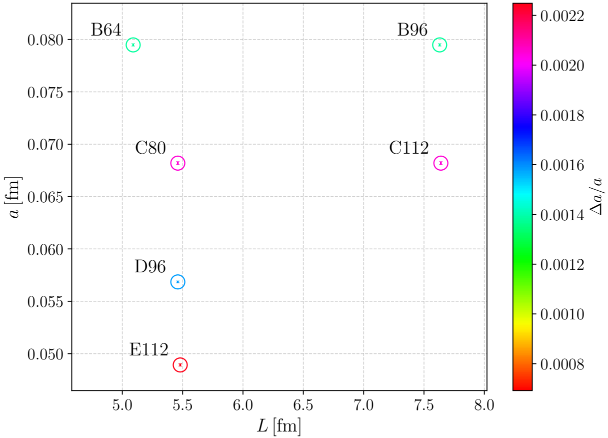
Outlook

- Fully consistent with the Fermilab measurement of a_μ^{exp} [Aliberti et al., 2025]; strengthens the tension with the dispersive (data-driven) determination
- Dedicated, first-principles calculation of isospin-breaking corrections is ongoing within ETMC
- Our result feeds into an updated lattice world average $\rightarrow$ uncertainty on the isospin-symmetric HVP reduced by $\simeq XX\%$
- Cross-checks built into the analysis: blinding, 3 independent groups, 2 discretizations (TM, OS), 2 isoQCD schemes (FLAG, WP25)

Thank You!

Backup

ETMC ensembles



Ensemble	a^{iso} [fm]
B64	0.07948(11)
B96	0.07948(11)
C80	0.06819(14)
C112	0.06819(14)
D96	0.056850(90)
E112	0.04892(11)

Statistics

Control variates with low-mode averaging

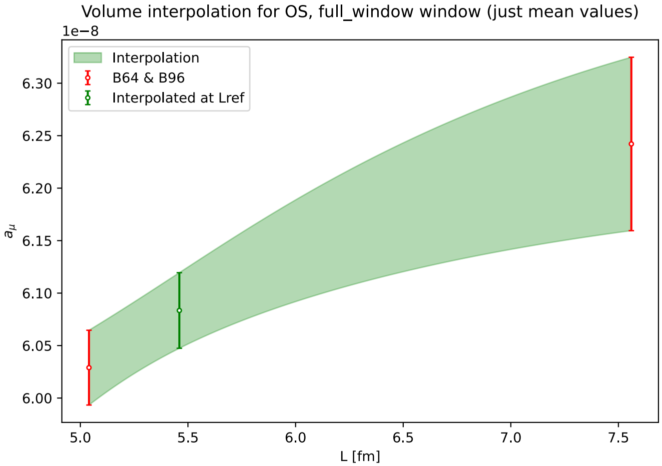
$$V_\eta^{\mathrm{LMA}}(t) = V_\eta(t) + \alpha(t)\Big[V^{\mathrm{IR}}_{\mathrm{impr.}}(t) - V^{\mathrm{IR}}_\eta(t)\Big],$$

$$\alpha(t) = \frac{\mathrm{Cov}\big[V_\eta(t), V_\eta^{\mathrm{IR}}(t)\big]}{\mathrm{Var}\big[V_\eta^{\mathrm{IR}}(t)\big]}, \qquad S^{\mathrm{IR}} = \sum_{j=0}^{N_{\mathrm{ev}}^{\mathrm{IR}}-1} \frac{1}{\lambda_j}\,\gamma_5 v_j v_j^\dagger,$$

Ensemble	N_U	$N_{\mathrm{ev}}^{\mathrm{IR}}$	N_η
B64	1300	400	1024
C80	590	530	1600
D96	570	530	960
E112	670	530	1344

Ensemble	N_U	N_η^{MG}	N_η
B96	490	11200	768

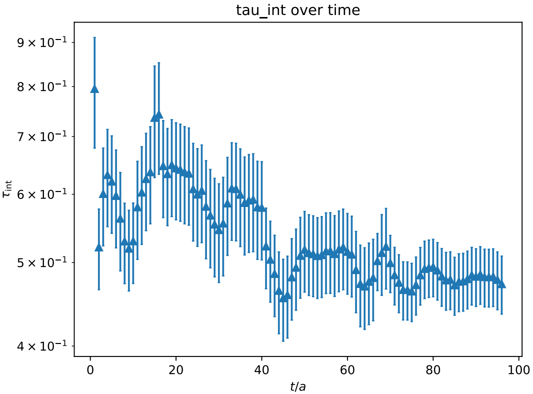
Volume interpolation to L_{ref}



Volume interpolation of $a_\mu^{\mathrm{HVP}}(I{=}1)$ to $L_{\mathrm{ref}} = 5.46$ fm (OS)

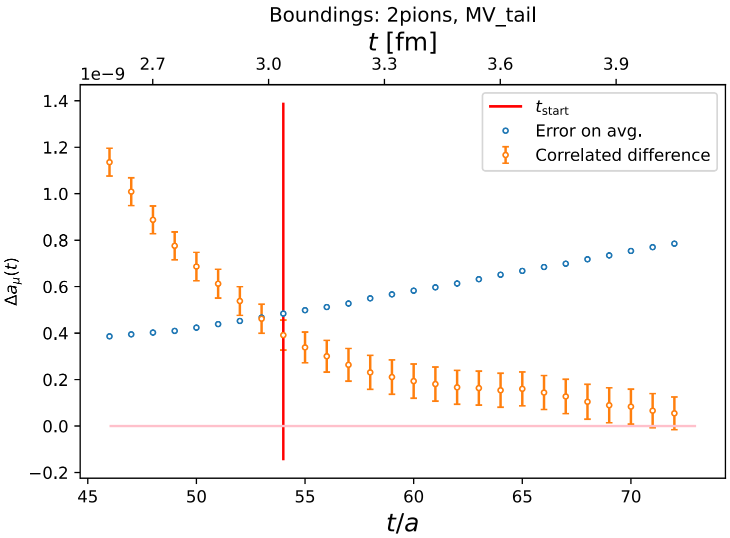
Bootstrap & binning

- Γ-method for autocorrelation [Wolff, 2004]
- Binning: $\tau_{\mathrm{int}} < 0.5$
- $N_b = 500$ bootstrap samples
- Trapezoidal TMR integration, bootstrap-by-bootstrap



$\tau_{\mathrm{int}}(t)$ for $V(t)$ on D96 (TM)

Bounding fit example: D96, TM, moderate



Continuum extrapolation: fit ansätze

4 spacings $\times$ 2 discretizations (TM, OS), correlated fit

$$\text{const: } a_\mu(a) = p_0$$
$$\text{lin: } p_0 + p_1\, a^2$$
$$\text{quad: } p_0 + p_1\, a^2 + p_2\, a^4$$
$$\text{Husung: } p_0 + p_1\, a^2\, \log^{\gamma_2}\!\left(\tfrac{1}{a\Lambda_{\mathrm{QCD}}}\right)$$
$$\text{lin+Husung: } p_0 + p_1\, a^2 + p_2\, a^2\, \log^{\gamma_2}\!\left(\tfrac{1}{a\Lambda_{\mathrm{QCD}}}\right)$$

- All share continuum value p_0 between TM & OS
- $\gamma_2 = 0.42$ [Husung, 2026], $\Lambda_{\mathrm{QCD}} = 0.3$ GeV
- AIC-weighted model average over all ansätz combinations

Error budget construction

Set of continuum-limit models, each with an AIC weight w_m:

1. For each systematic k with n_k choices, partition models by their value of k and collect bootstrap samples
2. Build a weighted histogram of a_μ for each of the n_k groups using the AIC weights
3. Apply the **law of total variance** at fixed k:

$$\sigma^2_{\text{tot}} = \underbrace{\mathbb{E}[\text{Var}(a_\mu|k)]}_{\text{stat. + other syst.}} + \underbrace{\text{Var}(\mathbb{E}[a_\mu|k])}_{\text{syst. from } k}$$

Equivalently, in the AIC-weighted form of [Alexandrou et al., 2024]:

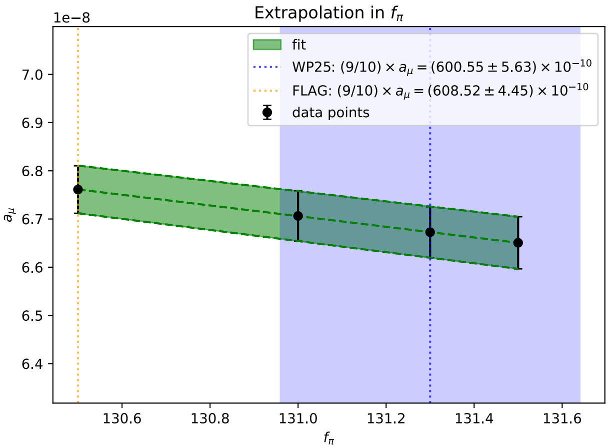
$$x = \sum_k \omega_k \, x_k \ , \quad \sigma^2_{\text{stat}} = \sum_k \omega_k \, \sigma^2_k \ , \quad \sigma^2_{\text{syst}} = \sum_k \omega_k \, (x_k - x)^2$$

Repeated for each systematic $k \to$ full error budget table. Model averaging & error budget handled by lattice_data_tools

Caveat

Individual systematics need not sum in quadrature (correlations between different k)

f_π dependence of $a_\mu^{\mathrm{HVP}}(I{=}1)$



Linear interpolation to WP25 target (preferred strategy)

History of ETMC simulations

Years	N_f	m_π	Reference(s)
2004 - 2010	2	> 200 MeV	[Jansen and Urbach, 2006]
2010 - 2014	$2+1+1$	> 200 MeV	[Carrasco et al., 2014]
2014 - 2018	2	$= 135$ MeV	[Abdel-Rehim et al., 2017]
2018 - 2024	$2+1+1$	$= 135$ MeV	[Alexandrou et al., 2021] [Alexandrou et al., 2023] [Alexandrou et al., 2024]

Remarks on fermions (ideal situation)

What we would like to have:

If we knew the bare masses exactly at the isoQCD physical point, we would like to simulate the probability distribution:

$$P[U] = e^{-S_{\mathrm{YM}}} \prod_f \mathcal{D}_f^{\mathrm{OS}}(a\mu_f^{\mathrm{phys}}, am_{0,f}^{\mathrm{phys}})$$

- $\mathcal{D} = \det D$: determinant of Dirac operator
- OS: Osterwalder-Seiler fermions (or any other type of fermions)

Remarks on fermions (ideal situation)

What we would like to have:

If we knew the bare masses exactly at the isoQCD physical point, we would like to simulate the probability distribution:

$$P[U] = e^{-S_{\rm YM}} \prod_f \mathcal{D}_f^{\rm OS}(a\mu_f^{\rm phys}, am_{0,f}^{\rm phys})$$

- $\mathcal{D} = \det D$: determinant of Dirac operator
- OS: Osterwalder-Seiler fermions (or any other type of fermions)

Problem

- We don't know a priori the exact values of the bare parameters

Remarks on fermions (practice)

What we actually have:

Despite being close to the isoQCD physical point, our probability measure is still:

$$P[U] = e^{-S_{\mathrm{YM}}} \prod_f \det D_f^{\mathrm{TM}}(a\mu_f^{\mathrm{sim}}, am_{0,f}^{\mathrm{sim}})$$

TM: Twisted Mass fermions [Shindler, 2008]

- Strictly positive fermionic determinant for $\mu \neq 0$
- If $am_0 = am_{cr} \to$ *maximal twist* $\to O(a)$ improvement

Remarks on fermions (practice)

What we actually have:

Despite being close to the isoQCD physical point, our probability measure is still:

$$P[U] = e^{-S_{\rm YM}} \prod_f \det D_f^{\rm TM}(a\mu_f^{\rm sim}, am_{0,f}^{\rm sim})$$

TM: Twisted Mass fermions [Shindler, 2008]

- Strictly positive fermionic determinant for $\mu \neq 0$
- If $am_0 = am_{cr} \to$ *maximal twist* $\to O(a)$ improvement

However, we need to correct the mistuning:

- $a\mu_f^{\rm sim} \neq a\mu_f^{\rm phys}$
- $am_{0,f}^{\rm sim} \neq am_{cr,f}$

Defining the mixed Action setup

Total action

All this is equivalent to a mixed action approach:

$$S = S_{\mathrm{YM}} + \sum_f (S_F)^{(f)}_{\mathrm{TM}} + S^{(f)}_{\mathrm{ghost}} + (S_F)^{(f),val}_{\mathrm{OS}}$$

For the gauge part we use the mean-field improved Iwasaki action [Iwasaki, 1985; S. Aoki, Frezzotti, and Weisz, 1999]:

$$S_{\mathrm{YM}} = \frac{\beta}{N_c} \sum_x \sum_{\mu<\nu} b_0 \left[1 - \mathrm{Re}\,\mathrm{Tr}\, U^{1\times 1}_{\mu\nu}(x)\right] + b_1 \left[1 - \mathrm{Re}\,\mathrm{Tr}\, U^{1\times 2}_{\mu\nu}(x)\right].$$

Fermions (sea contributions)

The Dirac operator for the sea quarks is the TM one:

- Light quarks (isosymmetric limit: $m_u = m_d = m_\ell$):

$$D_\ell^{\mathrm{TM}} = \gamma_\mu \nabla_\mu[U] - i\tau^3\gamma_5 \left(W^{cl}[U] + m_0\right) + \mu_\ell$$

- Heavy quarks:

$$D_h^{\mathrm{TM}} = \gamma_\mu \nabla_\mu[U] - i\tau^1\gamma_5 \left(W^{cl}[U] + m_0\right) + \mu_\sigma + \tau^3\mu_\delta$$

where $W^{cl}[U]$ is the the Wilson-Clover term [Sheikholeslami and Wohlert, 1985].

The action reads:

$$(S_F)_{\mathrm{TM}} = \sum_x \bar{\Psi}_\ell \, D_\ell^{\mathrm{TM}} \, \bar{\Psi}_\ell + \bar{\Psi}_h \, D_h^{\mathrm{TM}} \, \Psi_h$$

Fermions (sea and ghost pseudo-fermions)

The correction of the mistunings comes from both the sea and the valence.

OS fermions

The observables and the mistuning corrections are computed with OS fermions

$$D^f_{\mathrm{OS}} = \gamma_\mu \bar{\nabla}_\mu[U] - i r_f \gamma_5 \left(W^{\mathrm{cl}}[U] + m_0 \right) + m_f$$

where $r_{u,c} = 1$ and $r_{d,s} = -1$.

Fermions (sea and ghost pseudo-fermions)

The correction of the mistunings comes from both the sea and the valence.

OS fermions

The observables and the mistuning corrections are computed with OS fermions

$$D^f_{\mathrm{OS}} = \gamma_\mu \bar{\nabla}_\mu[U] - i r_f \gamma_5 \left(W^{\mathrm{cl}}[U] + m_0 \right) + m_f$$

where $r_{u,c} = 1$ and $r_{d,s} = -1$.

Applying the corrections in the sea is the same as:

- Including ghost pseudo-fermions through $S^{(f)}_{\mathrm{ghost}}$ $\rightarrow$ cancel simulated fermionic determinant
- Correct determinants from OS fermions at the isoQCD physical point and critical mass

Summary on Fermions

Corrections found with OS fermions $\rightarrow$ probability measure is:

$$\boxed{P[U] = \underbrace{e^{-S_{\mathrm{YM}}(g_0)}}_{\text{YM action}} \times \underbrace{\mathcal{D}^{\ell}_{\mathrm{TM}}(\mu_\ell^{\mathrm{sim}}, m_{\mathrm{cr}}^{\mathrm{sim}})\mathcal{D}^{h}_{\mathrm{TM}}(\mu_\sigma^{\mathrm{sim}}, \mu_\delta^{\mathrm{sim}}, m_{\mathrm{cr}}^{\mathrm{sim}})}_{\text{TM det. from sim}} \times \underbrace{\prod_f \frac{\mathcal{D}^f_{\mathrm{OS}}(m_f, m_{\mathrm{cr}})}{\mathcal{D}^f_{\mathrm{OS}}(m_f^{\mathrm{sim}}, m_{\mathrm{cr}}^{\mathrm{sim}})}}_{\text{Correction factor}}}$$

Matching TM and OS fermions [Frezzotti and Rossi, 2004]

$$m_\ell = \mu_\ell, \quad m_s = \mu_\sigma - \frac{Z_P}{Z_S}\mu_\delta, \quad m_c = \mu_\sigma + \frac{Z_P}{Z_S}\mu_\delta$$

- Light quarks: $\mathcal{D}_{\mathrm{TM}} = \mathcal{D}_{OS}$
- Heavy quarks: $\frac{\mathcal{D}^h_{TM}(\mu_\sigma^{\mathrm{sim}}, \mu_\delta^{\mathrm{sim}}, m_0^{\mathrm{sim}})}{\mathcal{D}^s_{OS}(m_s^{\mathrm{sim}}, m_0^{\mathrm{sim}})\cdot\mathcal{D}^c_{OS}(m_c^{\mathrm{sim}}, m_0^{\mathrm{sim}})} = 1 + O(a^2)$

Wrap-up on the mixed-action setup

(Reminder) What we want

$$P[U] = e^{-S_{YM}(g_0)} \times \prod_f \mathcal{D}^f_{OS}(m_f, m_0)$$

What we have

$$P[U] = e^{-S_{YM}(g_0)} \times \prod_f \mathcal{D}^f_{OS}(m_f, m_0) \times \left(1 + O(a^2)\right)$$

Wrap-up on the mixed-action setup

(Reminder) What we want

$$P[U] = e^{-S_{YM}(g_0)} \times \prod_f \mathcal{D}^f_{OS}(m_f, m_0)$$

What we have

$$P[U] = e^{-S_{YM}(g_0)} \times \prod_f \mathcal{D}^f_{OS}(m_f, m_0) \times \big(1 + O(a^2)\big)$$

What does it mean in practice?

1. Ensemble generation: good guesses for bare parameters
2. Find the slopes of observables with respect to sea and valence
3. Correct the bare parameters to reach the isoQCD physical point.

Diagrammatic representation of sea effects

Correction factor for sea effects

Sea quarks mistuning brings the following correction factor:

$$W_f(m_f, m_{cr}) = \hat{W}_f(m_{cr})\bar{W}_f(m_f)$$

$$\hat{W}_f(m_{cr}) = \frac{D^f_{OS}(m_0^{\rm sim}, m_{cr})}{D^f_{OS}(m_0^{\rm sim}, m_0^{\rm sim})}, \quad \bar{W}_f(m_f) = \frac{D^f_{OS}(m_f, m_{cr})}{D^f_{OS}(m_f^{\rm sim}, m_{cr})}$$

Expansion parameters

$$\Delta m_f = m_f - m_f^{\rm sim}, \quad \Delta m_{\rm cr} = m_{\rm cr} - m_0^{\rm sim}$$

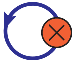
They are very small (both $\sim O(1\%)$) $\to$ we treat them as being of equal order Δm.

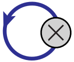
Diagrammatic representation of sea effects

Jacobi formula

For a parameter-dependent matrix $A(\lambda)$:

$$\partial_\lambda \det(A) = \det(A) \cdot \mathrm{Tr}\left(A^{-1}\partial_\lambda A\right)$$

$$\hat{W}_f(m_{cr}) = 1 + \Delta m_{\mathrm{cr}}\,\mathrm{Tr}\left[(-ir_f\gamma_5)\frac{1}{D_{OS}^{\mathrm{sim}}}\right] + O(\Delta m)^2 = 1 + \Delta m_{\mathrm{cr}} \cdot$$  $$+ O(\Delta m)^2$$

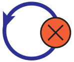
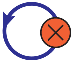
$$\bar{W}_f(m_f) = 1 + \Delta m_f\,\mathrm{Tr}\left[\frac{1}{D_{OS}^{\mathrm{sim}}}\right] + O(\Delta m)^2 = 1 + \Delta m_f \cdot$$  $$+ O(\Delta m)^2$$

Sea effects as a 1st order perturbation

$$S^{\text{phys}} = S^{\text{sim}} + \Delta S$$

$$\langle O \rangle^{\text{phys}} = \frac{\int \mathcal{D}U \, O[U] \, e^{-S^{\text{phys}}}}{\int \mathcal{D}U \;\, e^{-S^{\text{phys}}}} \approx \langle O \rangle^{\text{sim}} + \langle O \Delta S \rangle^{\text{sim}} - \langle O \rangle^{\text{sim}} \, \langle \Delta S \rangle^{\text{sim}}$$

Physical and critical mass corrections

$\partial^{\text{sea}}_{\text{cr},f} O[U] = \langle O[U]$  $\rangle^{\text{sim}} - \langle O[U] \rangle^{\text{sim}} \langle$  $\rangle^{\text{sim}}$

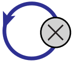
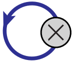
$\partial^{\text{sea}}_{f} O[U] = \langle O[U]$  $\rangle^{\text{sim}} - \langle O[U] \rangle^{\text{sim}} \langle$  $\rangle^{\text{sim}}$

Table of contents

Practical steps

- PCAC mass determination
- Correcting observables for $m_{\mathrm{PCAC}} \neq 0$
- Linear system on 3 observables for tuning Δm_f $(f = \ell, s, c)$

Tricks in the actual numerical calculation

Note: We use a set of tricks to ease the calculation:

- Analytic formulas for m_{PCAC} dependence of F_π and M_π
  $\rightarrow$ correction of $\Delta m_{\mathrm{cr}} \neq 0$ through m_{PCAC}
- Valence quark mass dependence by inversion of D_f at different values of the mass
  $\rightarrow$ slope found as finite difference

m_{PCAC} determination

$$2m_{\mathrm{PCAC}}^{\mathrm{sim}} = \frac{\sum_{\vec{x}} \langle \partial_0 \chi_\ell^\dagger \gamma_5 \gamma_0 \tau^1 \chi_\ell(t,\vec{x}) [\bar{\chi}_\ell \gamma_5 \tau^1 \chi_\ell](0) \rangle}{\sum_{\vec{x}} \langle [\bar{\chi}_\ell \gamma_5 \gamma_0 \tau^1 \chi_\ell](t,\vec{x}) [\bar{\chi}_\ell \gamma_5 \gamma_0 \tau^1 \chi_\ell](0) \rangle}$$

$$\chi_\ell(x) = \exp\left(-i\frac{\pi}{4}\gamma_5\tau^3\right)\Psi_\ell(x),$$

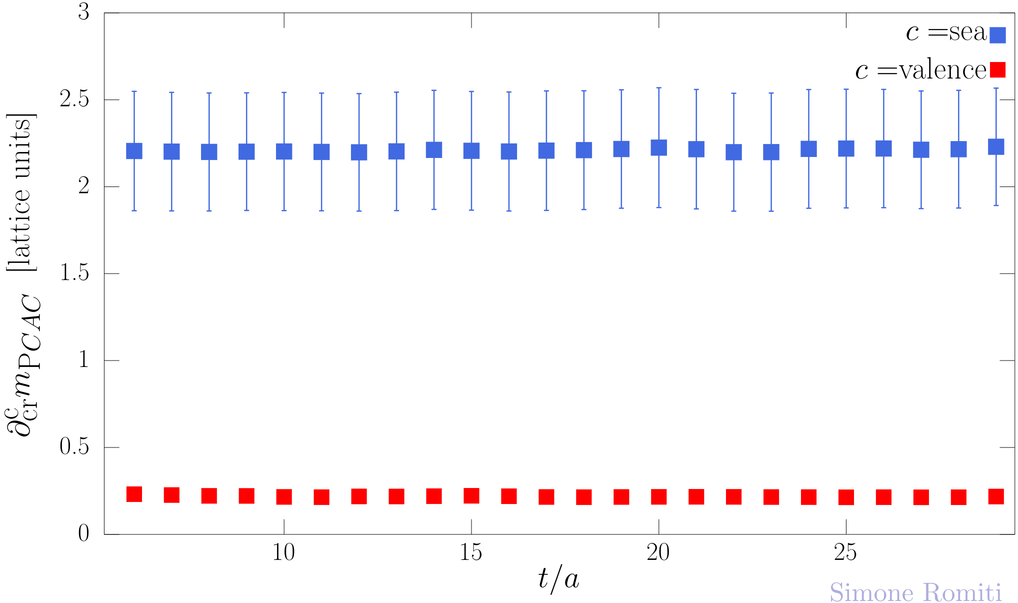
$$\bar{\chi}_\ell(x) = \bar{\Psi}_\ell(x)\exp\left(-i\frac{\pi}{4}\gamma_5\tau^3\right)$$

Twisted mass condition

$$0 = m_{\mathrm{PCAC}}^{\mathrm{sim}} + \Delta m_{\mathrm{cr}}\left[\partial_{cr}^{\mathrm{sea}} + \partial_{cr}^{\mathrm{val}}\right] m_{\mathrm{PCAC}}$$

$\rightarrow$ we find Δm_{cr}.

$m_{\rm PCAC}$ determination



m_{PCAC} dependence

The m_{PCAC} dependence of F_π and M_π is known [Alexandrou et al., 2021]:

$$F_\pi(m_{\mathrm{PCAC}} = 0) = F_\pi(m_{\mathrm{PCAC}})\sqrt{1 + \left(\frac{Z_A m_{\mathrm{PCAC}}}{m_\ell}\right)^2}$$

$$M_\pi^2(m_{\mathrm{PCAC}} = 0) = \frac{M_\pi^2(m_{\mathrm{PCAC}})}{\sqrt{1 + \left(\frac{Z_A m_{\mathrm{PCAC}}}{m_\ell}\right)^2}}$$

where Z_A is determined with the method of [Alexandrou et al., 2023].

$\rightarrow$ slopes with respect to m_{PCAC} are known.

Z_A determination

Tricks (see [Alexandrou et al., 2023]):

- In twisted mass, no need for renormalization constants for f_π:

$$f_\pi^{\mathrm{TM}} = 2(a\mu_\ell)\frac{|\langle 0|P_5^{\mathrm{TM}}(\ell)|\pi\rangle|}{M_\pi^{\mathrm{TM}}\sinh\left(aM_\pi^{\mathrm{TM}}\right)}$$

- Up to lattice artifacts, f_π is the same for both TM and OS:

$$f_\pi^{\mathrm{TM}} = f_\pi^{\mathrm{OS}} + O(a^2)$$

- Z_A found in the chiral limit:

$$Z_A(\mu_f) = Z_A(0) + O(a^2\mu_f^2)$$

Z_A determination

We extract Z_A from the ratio $R_A(t)$:

$$R_A(t) = 2\mu_\ell \frac{C_{\rm SS}^{\rm OS}}{\partial_t C_{\rm AS}^{\rm OS}} \frac{M_\pi^{\rm OS} \sinh\left(aM_\pi^{\rm OS}\right)}{M_\pi^{\rm TM} \sinh\left(aM_\pi^{\rm TM}\right)} \frac{|\langle 0|P_5^{\rm TM}(\ell)|\pi\rangle|}{|\langle 0|P_5^{\rm OS}(\ell)|\pi\rangle|} \rightarrow Z_A$$

Ensemble	Z_A of [Alexandrou et al., 2024]:
B64	0.74296(19)
B96	0.74261(19)
C80	0.75814(13)
C112	0.75824(15)
D96	0.77367(10)
E112	0.78548(9)

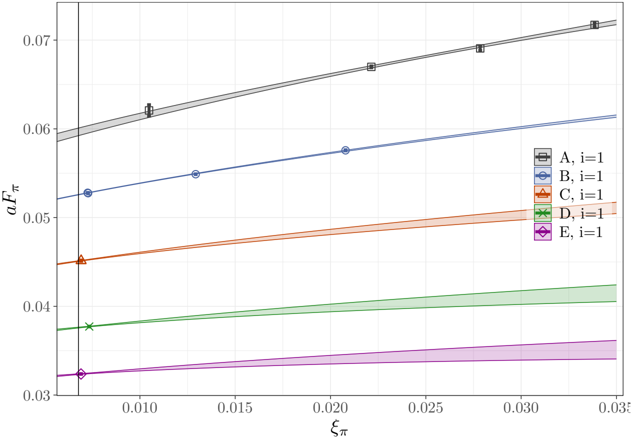
Light quark mass

F_π and M_π are more sensitive to m_ℓ than to m_s and m_c
$\to$ we follow 2 strategies to confirm the procedure

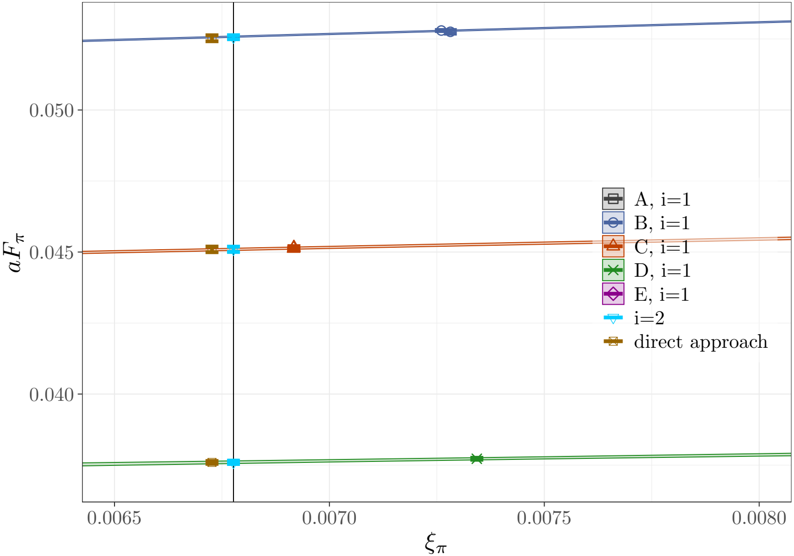
- *Direct approach*: rely on the linear approximation, and find the slopes as:
  1. $$\partial_{m_\ell{}^{\text{val}}} O = \frac{O(m_{\ell,2}) - O(m_{\ell,1})}{m_{\ell,2} - m_{\ell,1}}$$
  2. $$\partial_{m_\ell{}^{\text{sea}}} O = \langle O[U] \; \otimes \; \rangle^{\text{sim}} - \langle O[U] \rangle^{\text{sim}} \langle \; \otimes \; \rangle^{\text{sim}}$$
- *Global fit*: We fit aF_π and am_ℓ as a function of $\xi_\pi = \frac{M_\pi^2}{16\pi^2 F_\pi^2}$ (ansätze inspired by the Gasser-Leutwyler formulas from ChPT [Gasser and Leutwyler, 1984])

Global fit to determine a

$$aF_\pi(\xi_\pi,\beta) = a\overline{F_\pi}(\beta)\cdot\left\{1-2\xi_\pi\log\left(\frac{\xi_\pi}{\xi_\pi^{\mathrm{iso}}}\right)+\left[P+P_{\mathrm{disc}}\left(aF_\pi(\xi_\pi,\beta)\right)^2\right]\left(\xi_\pi-\xi_\pi^{\mathrm{iso}}\right)\right\}$$

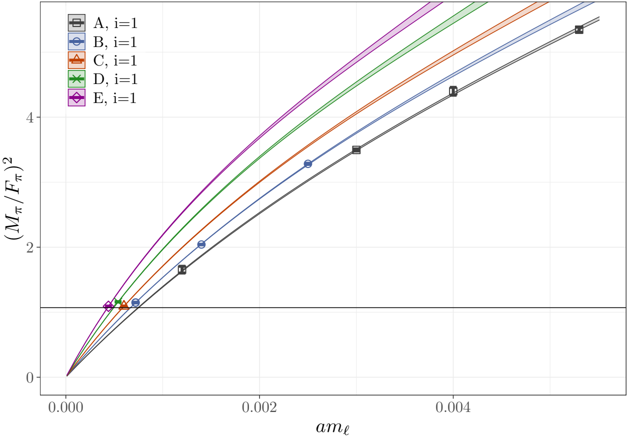


Global fit to determine a (zooming in)

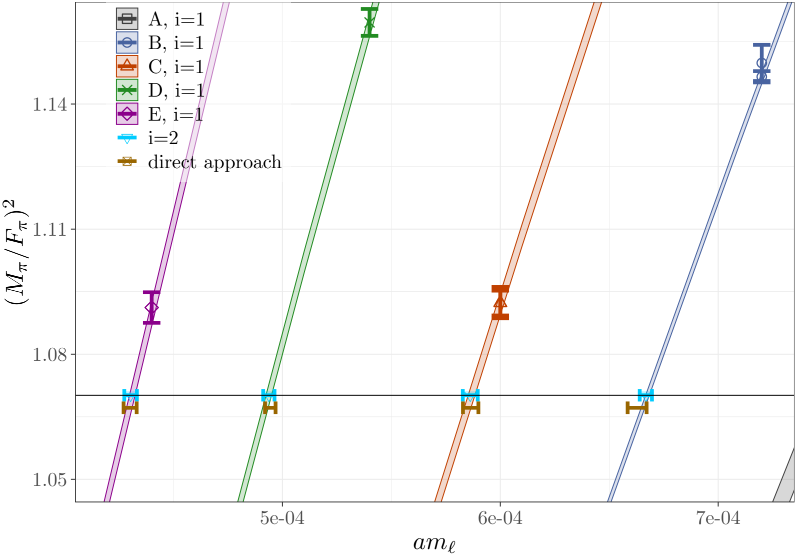


Global fit to determine m_ℓ^{phys}

$$am_\ell(\xi_\pi,\beta) = a\overline{m_\ell}(\beta)\frac{\xi_\pi}{\xi_\pi^{\mathrm{iso}}}\cdot\left\{1+5\xi_\pi\log\left(\frac{\xi_\pi}{\xi_\pi^{\mathrm{iso}}}\right)+\left(B+a^2B_{\mathrm{disc}}\right)\left(\xi_\pi-\xi_\pi^{\mathrm{iso}}\right)\right\}^{-1},$$

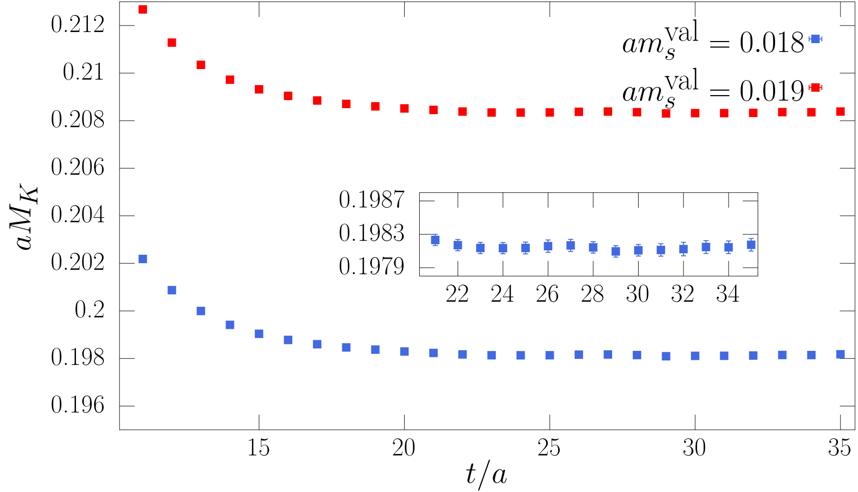


Global fit to determine m_ℓ^{phys} (zooming in)



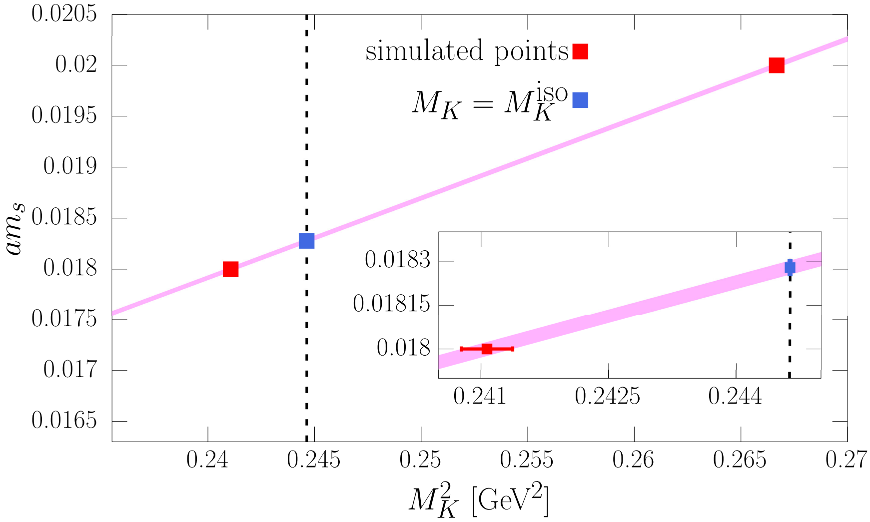
Strange quark mass

The Kaon mass is foud by fitting to a constant the effective mass curve



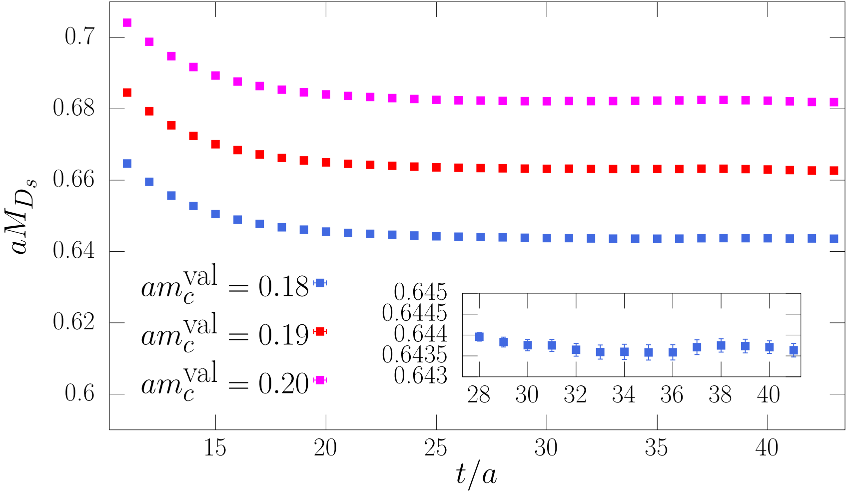
Strange quark mass

We interpolate using a simple linear ansatz: $m_s = A + BM_K^2$



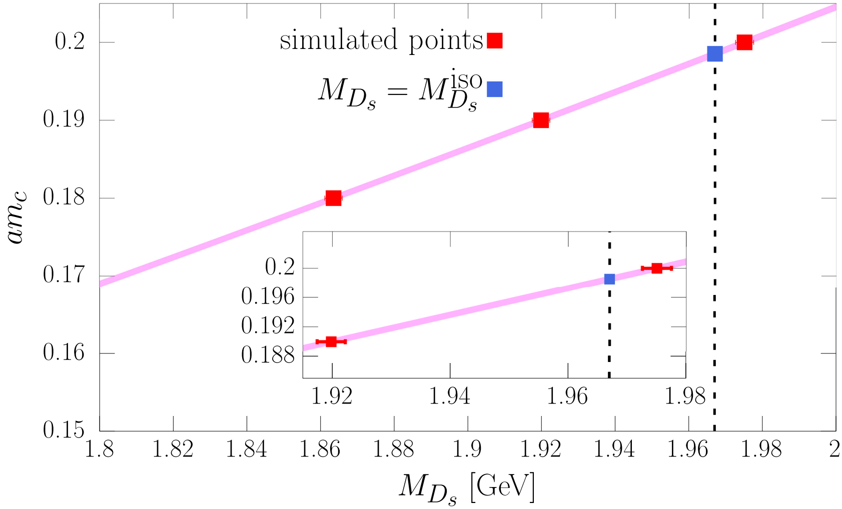
Charm quark mass

The D_s mass is found by fitting to a constant the effective mass curve:



Charm quark mass

We interpolate using a simple linear ansatz: $m_c = A + BM_D^2$



Setting the scale with the gradient flow

Setting the scale with the gradient flow (work in progress)

The gradient flow defines a smoothing map [Lüscher, Martin, 2010]:

$$\dot{U}_\mu(t,x) = D_\nu G_{\nu\mu}$$

- Flowed quantities are remormalized
- We can define the scale through (unphysical) values t_0, w_0

Definition of the scales [Borsányi et al., 2012]

$$t_0^2 \langle E(t = t_0) \rangle = 0.3$$

$$w_0^2 \frac{d}{dt} \langle t^2 E(t) \rangle |_{t=w_0^2} = 0.3$$

where $E(t) = 1 - \bar{P}(t)$ is the flowed action density

Setting the scale with the gradient flow (work in progress)

The scales t_0 and w_0 don't depend on the valence, **but they do on the sea.**

Sea corrections at Leading Order

$$\langle E(t)\rangle_{\text{phys}} \approx \langle E(t)\rangle_{\text{sim}} + \langle O\rangle^{\text{sim}} + \langle O\Delta S\rangle^{\text{sim}} - \langle O\rangle^{\text{sim}} \langle \Delta S\rangle^{\text{sim}}$$

In our setup, sea corrections come from both Δm_f and Δm_0:

$$\partial^{m_0} \langle E(t)\rangle = \left[\langle E(t) \cdot \circlearrowleft\otimes \rangle^{\text{sim}} - \langle E(t)\rangle^{\text{sim}} \langle \circlearrowleft\otimes \rangle^{\text{sim}}\right]$$

$$\partial^{m_f} \langle E(t)\rangle = \left[\langle E(t) \cdot \circlearrowleft\otimes \rangle^{\text{sim}} - \langle E(t)\rangle^{\text{sim}} \langle \circlearrowleft\otimes \rangle^{\text{sim}}\right]$$

QCD+QED at LO

QCD+QED at LO

We consider a background of pure QCD in isospin symmetry:

$$S_0 = \int d^4x \mathcal{L} = \int d^4x \mathcal{L}_{QCD}(u = d = \ell)$$

Isospin Breaking (IB)

IB splits the degenerate doublet. At LO we have 2 types of effects:

- strong IB : $\hat{m}_u \neq \hat{m}_d \rightarrow O(\frac{\hat{m}_d - \hat{m}_u}{\Lambda_{QCD}}) \sim O(1\%)$
- QED : $q_f = ee_f \neq 0 \rightarrow O(\alpha_{EM}) \sim O(1\%)$

RM123 method

At LO, the slopes can be calculated in the isosymmetric theory (isoQCD).
$\rightarrow$ Their linear combinations with appropriate counterterms give the observables in the full theory QCD+QED

QCD+QED at LO

$$\mathcal{L} = \mathcal{L}_0 - \Delta m_{ud}\,\bar{q}\tau_3 q + eA_\mu \bar{q}\gamma_\mu \left(\frac{\tau_3}{2} + \frac{1}{6}\right) q$$

QED correction (lattice) [Divitiis et al., 2013]

$$\begin{aligned}\partial_{e^2}[T\langle O(x_i)\rangle] &= \int d^4x_1 d^4x_2 D_{\mu\nu}(x_1|x_2)\, T\langle O(x_i) J_\mu(x_1) J_\nu(x_2)\rangle \\ &+ \int d^4x D_{\mu\mu}(x|x)\, T\langle O(x_i) T_\mu(x)\rangle\end{aligned}$$

- $T_\mu(x)$ is the tadpole operator (lattice artifact)
- e^2 renormalizes at higher orders $\rightarrow$ we can use $\alpha_{\mathrm{EM}} = 1/137\ldots.$

QED on the lattice

Possible QED formulations

- Compact formulation: $U_\mu(x) = e^{i\omega_a \tau_a} \rightarrow U_\mu(x) = e^{ie\phi} e^{i\omega_a \tau_a}$
- Non-compact formulation: e^2 expansion and $D_{\mu\nu}$ computed separately (cheaper) $\rightarrow$ need to regularize the $k = 0$ mode (e.g QED$_L$)

Renormalization in tmQCD+QED

$$
\begin{array}{lcl}
m_f & \rightarrow & \Delta m_f \, [\bar{\psi}_f \psi_f] \\
m_f^0 & \rightarrow & \Delta m_f^0 \, [\bar{\psi}_f \, i\gamma^5 \tau^3 \psi_f] \\
g_s & \rightarrow & \Delta g_s \, G_{\mu\nu} G_{\mu\nu}
\end{array}
$$

Remarks

- Δg_s is equivalent to a change in the lattice spacing
- Δm_f^0 preserves the *maximal twist* condition in QCD+QED

Phenomenological considerations

Conventions

- The separation of strong IB and QED effects is arbitrary, but **the total contribution has to be the same**
- It is useful to use the same scheme $\rightarrow$ check among collaborations

Physical remarks and intuitive understanding

- QED gives an electromagnetic mass according to q^2
- This would make the u heavier than the d, but $m_d - m_u > 0! \implies$ the intrinsic difference in mass in compensated by QED
- LIBEs are the cancellation of 2 effects of the same order of magnitude $\rightarrow$ lower signal to noise ratio

Remarks on tuning to maximal twist

Remarks on tuning to maximal twist

Maximal twist imposed from the PCAC Ward Identity:

$$0 \equiv am_f^{\mathrm{PCAC}} = \frac{\left\langle \partial_\mu A_1^{(f)}(x_\mu) P_1^{(f)}(0) \right\rangle}{2 \left\langle P_1^{(f)}(x) P_1^{(f)} \right\rangle}$$

Practical recipe

- The critical mass m_{cr} is flavor-independent up to $O(a)$ effects [Chiarappa et al., 2007]:
- It is sufficient to tune m_0 for the lightest quark mass to have $O(a^2)$ effects in the observables [Boucaud et al., 2008]
- In our simulations we require $\tan\omega > 10$, i.e. [Shindler, 2008]:

$$am_{\mathrm{PCAC}}(m_f, m_0) < 0.1 \cdot (am_\ell^{\mathrm{sim}})/Z_A$$

This ensures $O(1\%)$ effects on F_π and M_π.

Backup: uncertainty on the QED kernel

$K(m_\mu t)$ depends on the lattice spacing a through m_μ in lattice units $\rightarrow$ it also needs a bootstrap sample.

- Checked numerically: correlation between bootstrap samples of a and of $V(t)$ is negligible $\rightarrow$ generated independently
- Avoid recomputing the full kernel integral bootstrap-by-bootstrap: linearize around the (unbiased) mean lattice spacing $\bar{a}$

$$\boxed{K(m_\mu t) = K(\bar{a}m_\mu \cdot t^{\rm lat}) + (a - \bar{a}) \left.\frac{\partial K}{\partial a}\right|_{\bar{a}m_\mu t^{\rm lat}}}$$

A closed-form expression for $\partial K/\partial a$ (cf. Appendix A of the paper draft) is evaluated once per bootstrap sample of a.

Backup: generating correlated bootstrap samples

The mistuning corrections ($a\mu_\ell^{\mathrm{sea}}$, $a\mu_s^{\mathrm{sea}}$, $a\mu_c^{\mathrm{sea}}$, am_0^{sea}) and the lattice spacing a must be sampled *consistently correlated*.

- Estimate the correlation matrix ρ (better behaved numerically than the covariance C, since $\rho_{ij} \in [-1, 1]$ and admits a Cholesky decomposition even when the statistical estimate of C does not)
- Cholesky decomposition $\rho = LL^T$

$$\rho = S^{-1} C S^{-1} \,, \qquad S_{ij} = \sigma_i \delta_{ij} \,, \qquad C = (SL)(SL)^T$$

Correlated samples for variables x_i (mean $\bar{x}_i$, uncertainty σ_i):

$$x_i = \bar{x}_i + (SL)_{ij}\, z_j \,, \qquad z_j \sim \mathcal{N}(0, 1)$$

Bibliography I

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