Update on $a_\mu^{\mathrm{HVP,isoQCD}}$ from the ETM collaboration

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5 August 2026

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Introduction

$g_\mu - 2$: benchmark of precision physics [Aliberti et al., 2025]:

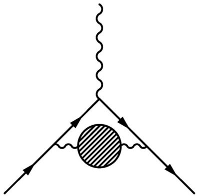
$$a_\mu^{\mathrm{SM}} = 116592033(62) \cdot 10^{-11}$$
$$a_\mu^{\mathrm{exp}} = 116592071.5(14.5) \cdot 10^{-11}$$

Hadronic contribution

Main source of uncertainty coming from:

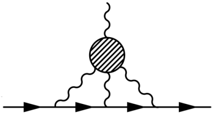
$$a_\mu^{\mathrm{had}} = a_\mu^{\mathrm{HVP}} + a_\mu^{\mathrm{HLbL}}$$

Hadronic Vacuum Polarization (HVP)



Hadronic Light by Light (HLbL)

See talk by Urs Wenger



isoQCD + QED

- **Theory**: The action S includes all the SM
- **Practice**: The HVP is dominated by **isoQCD** + QED (QCD only with $m_\ell = m_u = m_d$).
- **Numerical advantage**: Simulating isoQCD is cheaper, QED effects can be added later at 1st order

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Including Isosping Breaking effects

isoQCD precision reaches $O(1\%) \implies$ account for $m_u \neq m_d$ and $\alpha_{\mathrm{EM}} \neq 0$.

- *Leading Isospin Breaking Effects* (LIBEs): combination of 2 $O(1\%)$ effects

In this talk we focus on the isoQCD contribution to $a_\mu^{\mathbf{HVP}}$.

a_μ^{HVP} from the lattice

$$a_\mu^{\mathrm{HVP}} = 2\alpha_{em}^2 \int_0^\infty dt\, t^2\, K(m_\mu t)\, V(t)$$

$$K(z) = 2\int_0^1 dy(1-y)\left[1 - j_0^2\left(\frac{z}{2}\frac{y}{\sqrt{1-y}}\right)\right] , \qquad j_0(\xi) = \frac{\sin(\xi)}{\xi}$$

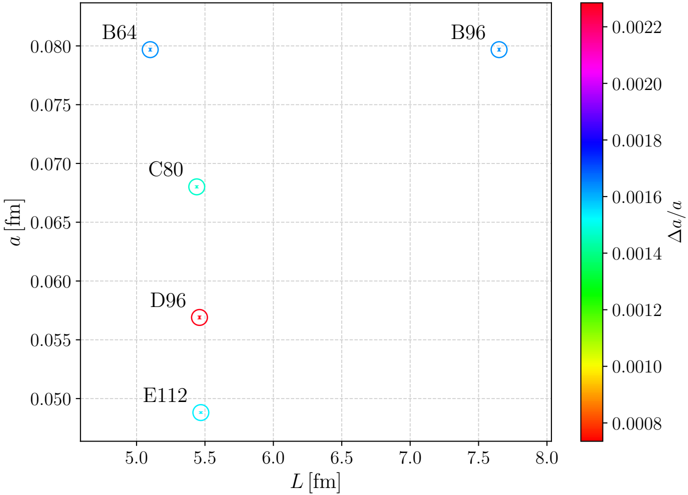
From the lattice

$$V(t) \equiv \frac{1}{3}\sum_{i=1,2,3}\int d^3x\, \langle J_i(\vec{x},t) J_i^\dagger(0)\rangle \qquad \text{(Wick contractions)}$$

with $J_i(x)$ being the electromagnetic current operator

$$J_i(x) \equiv \sum_{f=u,d,s,c,\ldots} q_{\mathrm{em},f}\, j_i^f(x) \qquad j_i^f(x) = \overline{\psi}_f(x)\gamma_i\psi_f(x) \qquad i = 1,2,3$$

ETMC ensembles (FLAG scheme)



Ensemble	a^{iso} [fm]
B64	0.07968(13)
B96	0.07968(13)
C80	0.06801(10)
D96	0.05690(13)
E112	0.048809(75)

Ensemble	L [fm]
B64	5.10
B96	7.65
C80	5.44
D96	5.46
E112	5.47

Fermion setup

ETMC uses **twisted-mass** (TM) fermions [Frezzotti, Grassi, et al., 2001] for sea quarks:

$$\boxed{D_\ell^{\mathrm{TM}} = \gamma_\mu \nabla_\mu - i\gamma_5 \tau^3 \left(W_{\mathrm{cl}} + m_0\right) + \mu_\ell}$$

Advantages of twisted mass [Frezzotti and Rossi, 2004a]

- Strictly positive determinant for $\mu \neq 0$
- $m_0 = m_{\mathrm{cr}} \to$ **maximal twist** $\to$ automatic $O(a)$ improvement

TM and OS regularizations for connected contributions

- **TM**: valence TM fermions, renormalization with Z_A
- **OS** (Osterwalder–Seiler): valence OS fermions, renormalization with Z_V

$V_{\mathrm{TM}}(t) - V_{\mathrm{OS}}(t) = O(a^2)$, common continuum limit [Frezzotti and Rossi, 2004b]

Definition of the isoQCD physical point

$$R_\pi = (M_\pi/f_\pi)^2 \quad R_K = (M_K/f_\pi)^2 \quad R_{D_s} = (M_{D_s}/f_\pi)$$

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Edinburgh/FLAG
consensus ["Converging on
QCD+QED prescriptions" 2023;
Y. Aoki et al., 2024]

$$f_\pi^{\text{iso}} = 130.5 \text{ MeV} , \quad a = \frac{(a f_\pi)}{f_\pi^{\text{iso}}}$$

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WP25 scheme

The scale is set through w_0 [Aliberti et al., 2025]:

$$w_0^{\text{iso}} = 0.17236\,\text{fm}\,, \qquad a = \frac{w_0^{\text{iso}}}{(w_0/a)}$$

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In both schemes:

$$M_\pi^{\rm iso} = 135.0\ \text{MeV}, \quad M_K^{\rm iso} = 494.6\ \text{MeV}, \quad M_{D_s}^{\rm iso} = 1967\ \text{MeV}.$$

RM123-like method for mistuning corrections [de Divitiis et al., 2013]

Remarks

- Our bare parameters do not match exactly the FLAG scheme
- and we do not want to redo the simulation
- but we are very close already to FLAG scheme. Can we use this?

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First-order Taylor expansion

Expand observable O around simulated point $\hat{m}^{\mathrm{sim}}$:

$$\boxed{O(m) = O^{\mathrm{sim}} + \sum_f \Delta\hat{m}_f \left.\frac{\partial O}{\partial m_f}\right|_{\mathrm{sim}}}$$

where mass mistunings are $\Delta\hat{m}_f = \hat{m}_f - \hat{m}_f^{\mathrm{sim}}$.

Derivatives via (pseudo)scalar density insertions

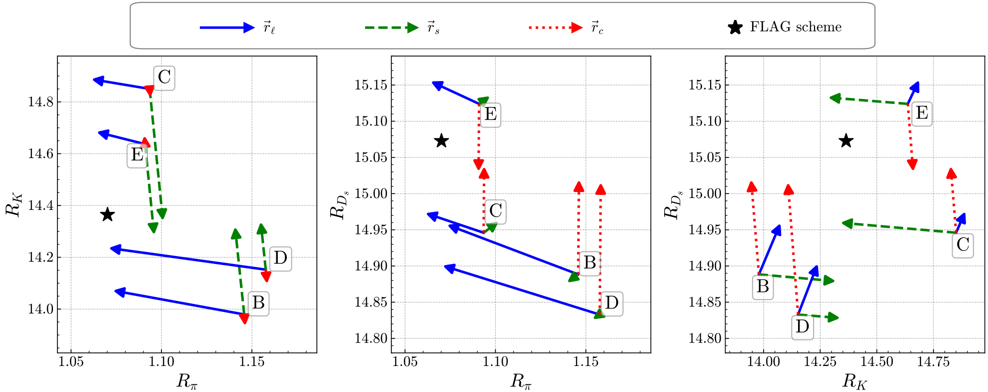
$\mathcal{J}_f = \int d^4x\, \bar{\psi}_f \Gamma \psi_f(x)$ treated as a perturbation:

$$\boxed{\frac{\partial\langle O\rangle}{\partial m_f} = -\Big[\langle O\,\mathcal{J}_f\rangle - \langle O\rangle\langle\mathcal{J}_f\rangle\Big]_{\mathrm{sim}}}$$

Avoids re-generating gauge configurations.

Tuning to the isoQCD physical point

$$\vec{R}^{\rm sim} + \sum_{f=\ell,s,c} (m_f - m_f^{\rm sim}) \frac{\partial \vec{R}}{\partial m_f}\Big|_{\rm sim} + (m_0 - m_0^{\rm sim}) \frac{\partial \vec{R}}{\partial m_0}\Big|_{\rm sim} = \vec{R}^{\rm FLAG}$$



From the mistuning solution to a_μ^{HVP}

1. Δm_0: preserving maximal twist

$$\boxed{m_{\mathrm{PCAC}}^{\mathrm{sim}} + (m_0 - m_0^{\mathrm{sim}}) \frac{dm_{\mathrm{PCAC}}}{dm_0}\bigg|_{\mathrm{sim}} = 0}$$

From the mistuning solution to a_μ^{HVP}

1. Δm_0: preserving maximal twist

$$m_{\mathrm{PCAC}}^{\mathrm{sim}} + (m_0 - m_0^{\mathrm{sim}}) \frac{dm_{\mathrm{PCAC}}}{dm_0}\bigg|_{\mathrm{sim}} = 0$$

2. Correcting $V(t)$ and a_μ^{HVP}

$$\frac{\partial}{\partial \hat{m}} a_\mu^{\mathrm{HVP}} = 2\alpha_{em}^2 \int_0^\infty dt\, t^2 K(m_\mu t)\, \frac{\partial}{\partial \hat{m}} V(t)\,, \qquad \hat{m} = m_\ell, m_s, m_c, m_0$$

$$a_\mu^{\mathrm{isoQCD}} = a_\mu^{\mathrm{sim}} + \sum_{f=\ell,s,c} \Delta m_f \frac{\partial a_\mu^{\mathrm{HVP}}}{\partial m_f}\bigg|_{\mathrm{sim}} + \Delta m_0 \frac{\partial a_\mu^{\mathrm{HVP}}}{\partial m_0}\bigg|_{\mathrm{sim}}$$

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Statistics

Control variates with low-mode averaging

$$V_\eta^{\mathrm{LMA}}(t) = V_\eta(t) + \alpha(t)\left[V_{\mathrm{impr.}}^{\mathrm{IR}}(t) - V_\eta^{\mathrm{IR}}(t)\right],$$

$$\alpha(t) = \frac{\mathrm{Cov}\left[V_\eta(t), V_\eta^{\mathrm{IR}}(t)\right]}{\mathrm{Var}\left[V_\eta^{\mathrm{IR}}(t)\right]}$$

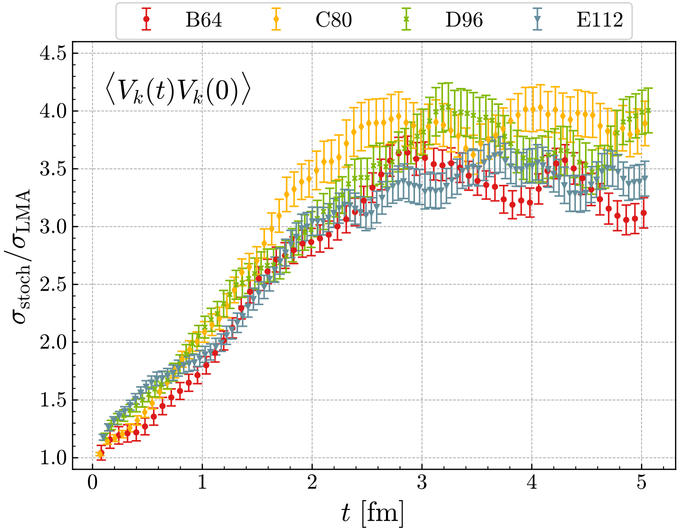
$$(D^{-1})^{\mathrm{IR}} = \sum_{j=0}^{N_{\mathrm{ev}}^{\mathrm{IR}}-1} \frac{1}{\lambda_j} \gamma_5 v_j v_j^\dagger,$$

(λ_i, v_i) eigenvalues and eigenvectors of $D\gamma_5$.

Ensemble	N_U	$N_{\mathrm{ev}}^{\mathrm{IR}}$	N_η
B64	1300	400	1024
C80	590	530	1600
D96	570	530	960
E112	670	530	1344

Ensemble	N_U	N_η^{MG}	N_η
B96	490	11200	768

Low Mode Averaging (LMA)



Isospin decomposition: isoscalar vs isovector

Flavor decomposition reorganized in definite isospin:

$$J_\mu^{I=1}(x) = \frac{1}{2}\left[j_\mu^u(x) - j_\mu^d(x)\right] \qquad J_\mu^{I=0}(x) = \frac{1}{6}\left[j_\mu^u(x) + j_\mu^d(x)\right] - \frac{1}{3}j_\mu^s(x) + \frac{2}{3}j_\mu^c(x)$$

The EM current separates as $J_\mu^{\text{em}} = J_\mu^{I=1} + J_\mu^{I=0}$, where the isoscalar $J^{I=0}$ contains the remaining flavor-diagonal contributions.

$$\boxed{a_\mu^{\text{HVP}}(I{=}1) = \frac{9}{10}\, a_\mu^{\text{HVP}}(\ell)\,, \qquad a_\mu^{\text{HVP}}(I{=}0) = \frac{1}{10}\, a_\mu^{\text{HVP}}(\ell) + a_\mu^{\text{HVP}}(s + c + \text{disc.})}$$

Details on the isospin separation [Kuberski, 2024]

- $I = 1$ and $I = 0$ have different leading pion states in the tail
- $V^{I=1}(t)$, $V^{I=0}(t)$ separately positive definite $\rightarrow$ bounding method applicable per channel

The $I = 0$ channel

Convenient to reorganize using the octet current [D. Djukanovic et al., 2025]

$$J^8_\mu(x) = \frac{1}{6}\Big[\big(j^u_\mu(x) + j^d_\mu(x)\big) - 2j^s_\mu(x)\Big]$$

$$\boxed{a_\mu^{\mathrm{HVP}}(I{=}0) = a_\mu^{\mathrm{HVP}}(\mathrm{oct.}) + a_\mu^{\mathrm{HVP}}(c) + \delta a_\mu^{c,\mathrm{disc}}}$$

- $a_\mu^{\mathrm{HVP}}(\mathrm{oct.})$: from J^8_μ, disconnected light + strange loops, OS only
- $a_\mu^{\mathrm{HVP}}(c)$ and $\delta a_\mu^{c,\mathrm{disc}}$: work in progress

In the paper we will provide an update for $a_\mu^{\mathrm{HVP}}(s)$ and $a_\mu^{\mathrm{HVP}}(c)$ with respect to [Alexandrou et al., 2024].

Octet contribution: one-end trick (OET)

Decomposition

$$a_\mu^{\mathrm{HVP}}(I{=}0) = a_\mu^{\mathrm{HVP}}(\mathrm{oct.}) + a_\mu^{\mathrm{HVP}}(c) + \delta a_\mu^{c,\mathrm{disc}}$$

Octet current, disconnected (one-end trick):

$$j_\mu^{(8)}(x) = \tfrac{1}{6}\left[j_\mu^u(x) + j_\mu^d(x) - 2\, j_\mu^s(x)\right]$$

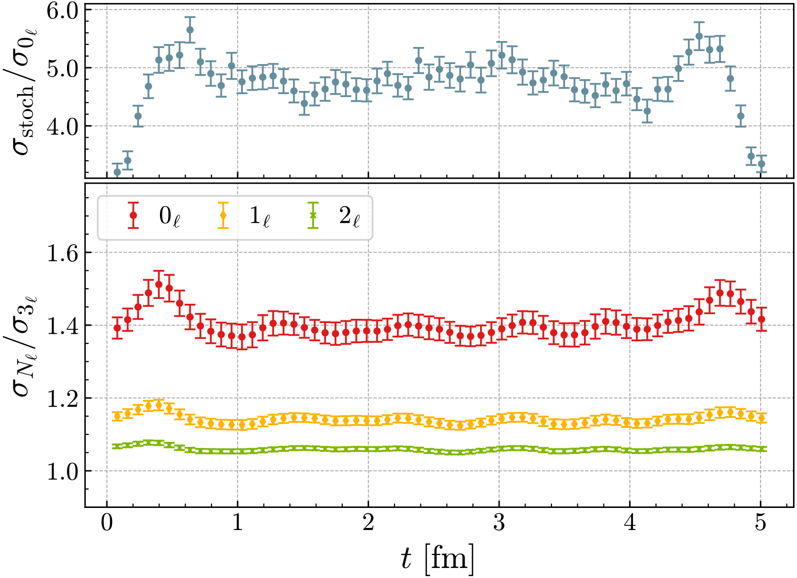
Frequency splitting [Giusti et al., 2019]:

$$\langle j_\mu^{(8)}(x)\rangle_\psi = B_\mu(m_\ell) - B_\mu(m_s) = [B_\mu(m_\ell) - B_\mu(m_1)] + \cdots + [B_\mu(m_N) - B_\mu(m_s)]$$

WTI:

$$B_\mu(m_\ell) - B_\mu(m_s) = \frac{1}{3}(\mu_\ell + \mu_s)\int d^4x\, \langle P^1(x)\, A_\mu^2(z)\rangle_\psi$$

OET and frequency splitting



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Blinding procedure

Exponential signals added to $V_\ell(t)$ **before** analysis:

$$V_\ell(t) \to V_\ell(t) + \frac{1}{Z^2} \sum_{i=1}^{k_{\text{group}}} |A_i|^2 \left[e^{-E_i t} + e^{-E_i(T-t)}\right]$$

Shift identical on all lattice spacings.

Blinding

- 3 independent groups: **A, B, C**
- $\{A_i, m_i\}$ unknown to each group
- $k_A = 14$, $k_B = 18$, $k_C = 16$ number of signals
- Random energies in the range:

$$0.75\,\mathrm{GeV} \leq E_i \leq 2\,\mathrm{GeV}$$

Bounding method [Lehner, 2016; Borsanyi et al., 2017]

Signal to noise ratio

$V(t)$ for large t: approaching 0 and increasing the noise.
If we integrate up to T, we mostly integrate noise $\rightarrow$ low precision on a_μ^{HVP}
$\rightarrow$ **We use an upper and lower bound, that meet before signal degradation**

$$\boxed{V^X(t_c)\, e^{-m_{\mathrm{eff}}(t_*)(t-t_c)} \;\leq\; V^X(t) \;\leq\; V^X(t_c)\, e^{-E_0^X\,(t-t_c)}}$$

- **Upper bound:** 2(3)-pions state on the lattice (final state interactions via GS-model estimates)
- **Lower bound:** Use m_{eff} for $t > t_{\mathrm{thr}}$

Bounding method of $I = 1$ and $I = 0$ contributions

$I = 1$: conservative lower bound for $\pi\pi$ state

- Attractive interaction in $\pi\pi$ states
- Lowest $\pi\pi$ state with Gounaris-Sakurai model is lowered by less than 2%

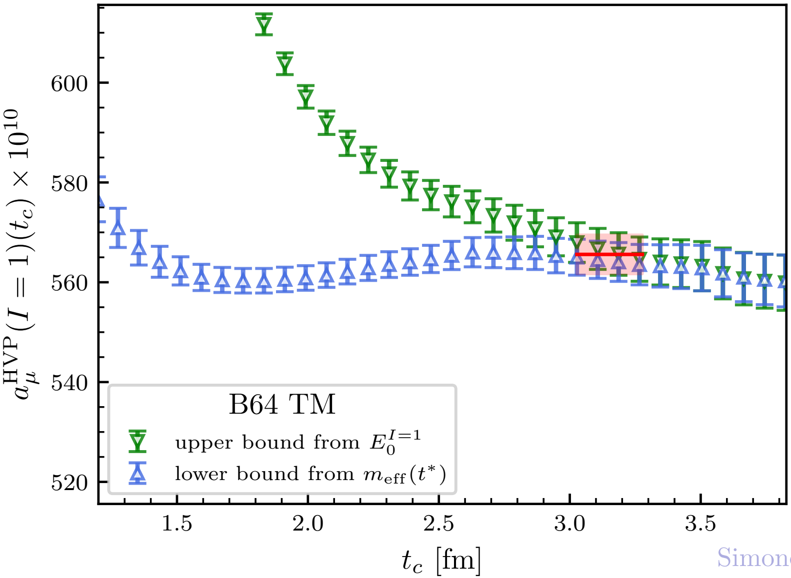
$$E_{0,\text{free}}^{I=1}\,(\text{OS}) = 0.98 \cdot 2\sqrt{m_\pi^2 + \left(\tfrac{2\pi}{L}\right)^2}$$

$$E_{0,\text{free}}^{I=1}\,(\text{TM}) = 0.98 \left[\sqrt{m_\pi^2 + \left(\tfrac{2\pi}{L}\right)^2} + \sqrt{m_{\pi^0}^2 + \left(\tfrac{2\pi}{L}\right)^2}\right]$$

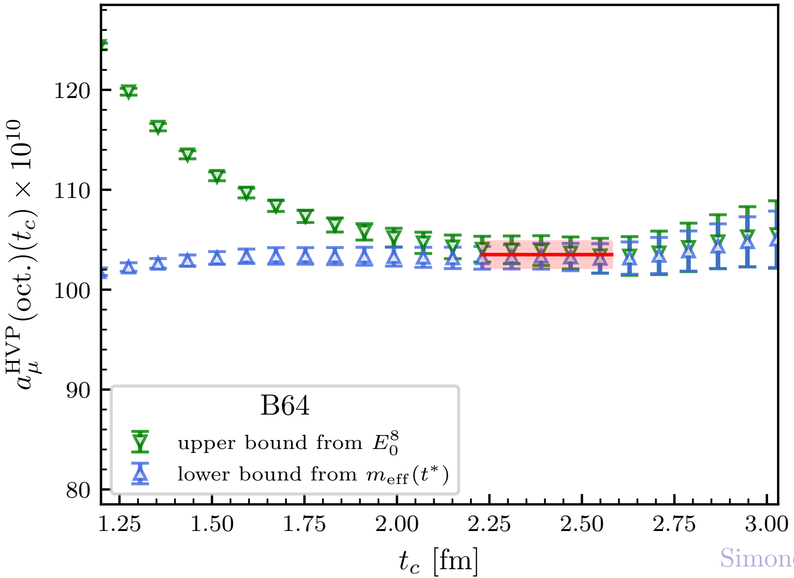
$I = 0$: 3π state

$$E_0^8 = 2\sqrt{m_\pi^2 + \left(\tfrac{2\pi}{L}\right)^2} + m_{\pi^0}$$

Bounding method ($I = 1$)



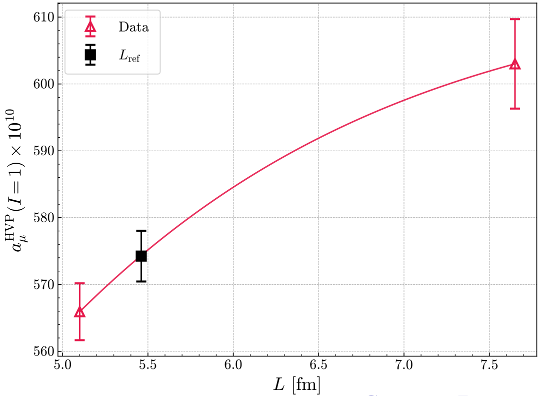
Bounding method for the octet



Reference-volume interpolation

$$a_\mu^{\mathrm{HVP}}(I{=}1;L) = a_\mu^{\mathrm{HVP}}(I{=}1;\infty) + ALe^{-M_\pi L}$$

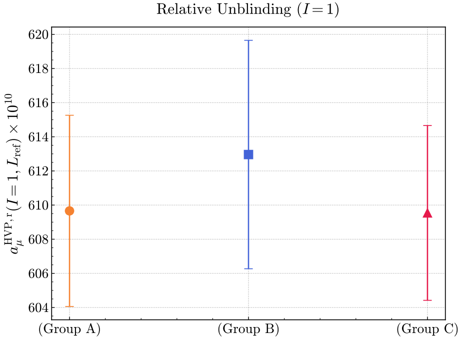
- Two volumes at coarsest spacing: **B64**, **B96**
- Interpolate to common L_{ref}
- Continuum extrapolation at fixed L_{ref}
- A determined from the B64–B96 difference at fixed a



Class of continuum fit ansätze

$$a_\mu^{\mathrm{HVP},r}(a, L_{\mathrm{ref}}) = a_\mu^{\mathrm{HVP}}(L_{\mathrm{ref}}) + c_1^r\, a^2 + c_{1L}^r \frac{a^2}{\left|\log(a^2\Lambda_0^2)\right|^\gamma} + c_2^r\, a^4\,, \qquad r = \mathrm{TM, OS}$$

- **Reference volume**: $L_{\mathrm{ref}} = 5.46$ fm
- Several variants from removing terms from the fit function
- $\gamma_2 = 0.42$, based on [Husung, 2025] and extrapolation with $N_f = 4$
- Bayesian/AIC model average (BAIC)



BAIC model averaging

The continuum limits are averaged according to the Bayesian Akaike Information Criterion:

$$\tilde{w}_i = \exp\left[-(\chi^2 + 2N_{\text{par}} - 2N_{\text{data}})/2\right], \quad w_i = \tilde{w}_i / \sum_i \tilde{w}_i$$

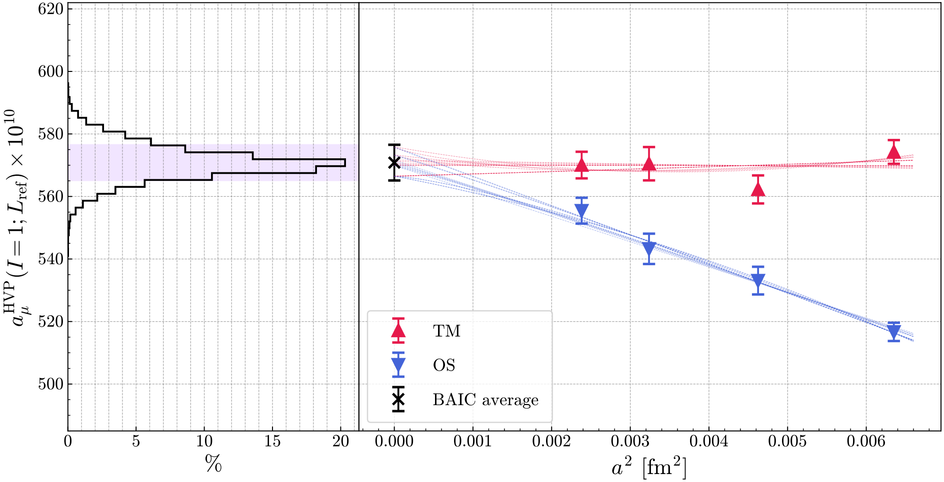
Law of total variance

$$\boxed{\sigma^2_{\text{tot}} = \underbrace{\mathbb{E}[\text{Var}(a_\mu)]}_{\text{stat.}} + \underbrace{\text{Var}(\mathbb{E}[a_\mu])}_{\text{syst.}} = \sum_m w_m \sigma^2_m + \sum_m w_m (a_{\mu,m} - \bar{a}_\mu)^2}$$

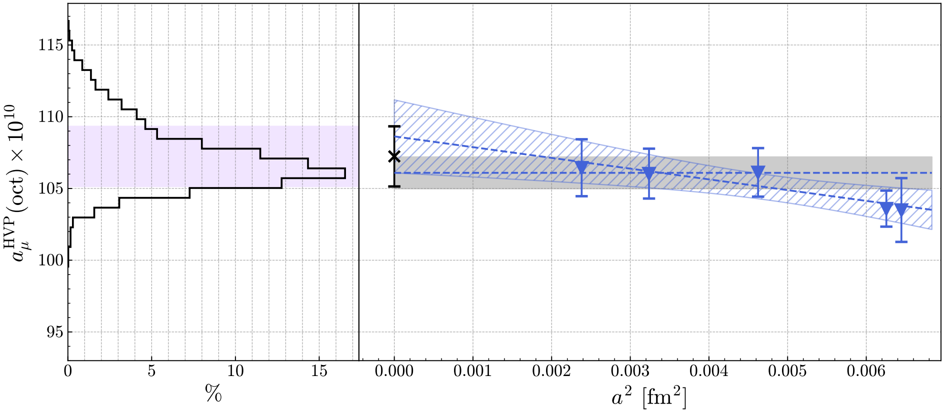
- **Estimator of the mean** (AIC-weighted):

$$\boxed{\bar{a}_\mu = \sum_{m \in \mathcal{M}} w_m a_{\mu,m}}$$

Continuum extrapolation for $I = 1$ channel



Octet contribution: continuum extrapolation



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Finite volume representation of $V^{I=1}(t, L)$

Two-pion contribution at finite volume

$$V_{\pi\pi}(t, L) = \sum_{n=1}^{\infty} \nu_n |A_n|^2 \, e^{-\omega_n t} \,, \qquad \omega_n = 2\sqrt{m_\pi^2 + k_n^2}$$

- Lüscher quantization condition:

$$\delta_{11}(k_n) + \phi(q_n) = n\pi \,, \qquad q_n = \frac{k_n L}{2\pi}$$

$$\nu_n |A_n|^2 = \frac{2k_n^5}{3\pi\omega_n^2} \frac{|F_\pi(\omega_n^2)|^2}{k_n \delta_{11}'(k_n) + q_n \phi'(q_n)}$$

Pion form factor: Gounaris–Sakurai parametrization

Validated by [Dalibor Djukanovic et al., 2025] for $t \geq 1.1\,\text{fm}$.

Adapting to twisted mass: $\pi^+\pi^0$ kinematics

- TM current $\to$ charged-current correlator $\Rightarrow$ lowest state is $\pi^+\pi^0$, not $\pi\pi$
- At finite a: $m_{\pi^0} \neq m_{\pi^+}$ (twisted-mass isospin breaking)

$$\boxed{E_{\pi^+}(k) = \sqrt{m_{\pi^+}^2 + k^2}\,, \qquad E_{\pi^0}(k) = \sqrt{m_{\pi^0}^2 + k^2}\,, \qquad \omega(k) = E_{\pi^+}(k) + E_{\pi^0}(k)}$$

Same dynamics, unequal-mass kinematics

- Same GS ansatz, same M_ρ, $g_{\rho\pi\pi}$
- Usual finite volume representation recovered exactly as $m_{\pi^+} \to m_{\pi^0}$

Residual FV correction for $I = 1$: $L_{\text{ref}} = 5.46$ fm $\to \infty$

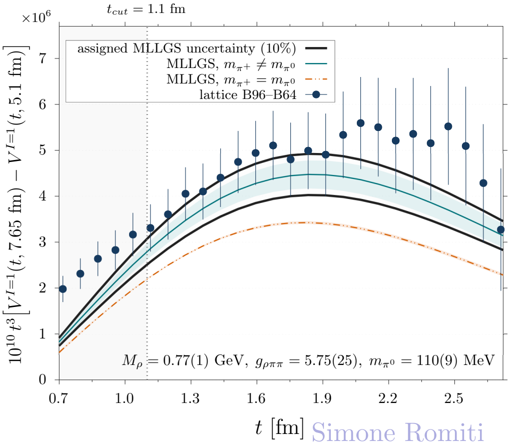
MLLGS model (adapted to twisted mass) [Harvey B. Meyer, 2011; Lellouch and Luscher, 2001; Luscher, 1991; Gounaris and Sakurai, 1968]

- TM fermions: $\pi^+\pi^0$ **dominated**

$$\Delta a_\mu^{\text{HVP}}(L) = a_\mu^{\text{HVP}}(\infty) - a_\mu^{\text{HVP}}(L)$$
$$= 27.1(2.5)(1.0)[2.7] \times 10^{-10}$$

FVE uncertainty given by:

- 10% on the MLLGS integrated curve above 1.1 fm.
- Integrated difference of B96 and B64 below 1.1 fm.



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Moving between schemes: two strategies

Direct w_0 input

- Solve mistuning system with w_0 instead of f_π
- × Larger cutoff effects

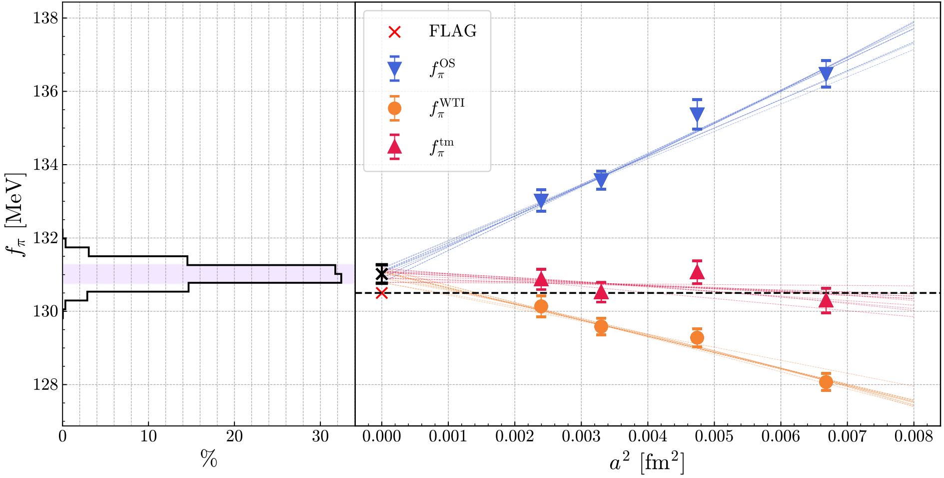
f_π-based matching

- Determine f_π^{WP25} equivalent to w_0^{WP25}
- Compute $a_\mu^{\mathrm{HVP}}(f_\pi)$ on neighboring schemes
- Interpolate to f_π^{WP25}

We adopt the f_π-based strategy

$$a_\mu^{\mathrm{HVP,\ WP25}}(f_\pi) = a_\mu^{\mathrm{HVP,FLAG}} + \left.\frac{\partial a_\mu^{\mathrm{HVP}}}{\partial f_\pi}\right|_{\mathrm{FLAG}} \left(f_\pi^{\mathrm{WP25}} - f_\pi^{\mathrm{FLAG}}\right)$$

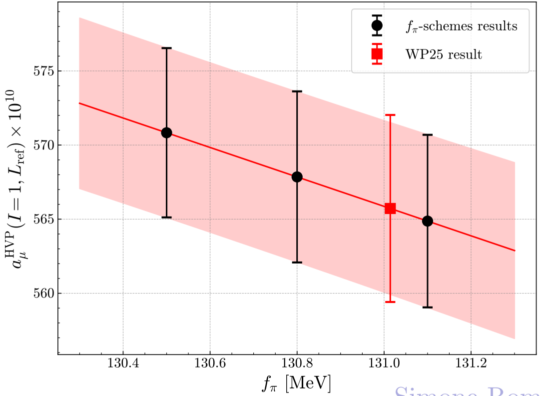
Finding f_π in the WP25 scheme



Moving from one scheme to the other

$$\frac{\partial a_\mu^{\mathrm{HVP}}(I{=}1)}{\partial f_\pi} = -9.9(2.2)\times 10^{-10}\,\mathrm{MeV}^{-1} \qquad \frac{\partial a_\mu^{\mathrm{HVP}}(\mathrm{oct.})}{\partial f_\pi} = -1.4(1.2)\times 10^{-10}\,\mathrm{MeV}^{-1}$$

- Redo the whole scale setting for several values of f_π
- The final result is obtained by interpolation



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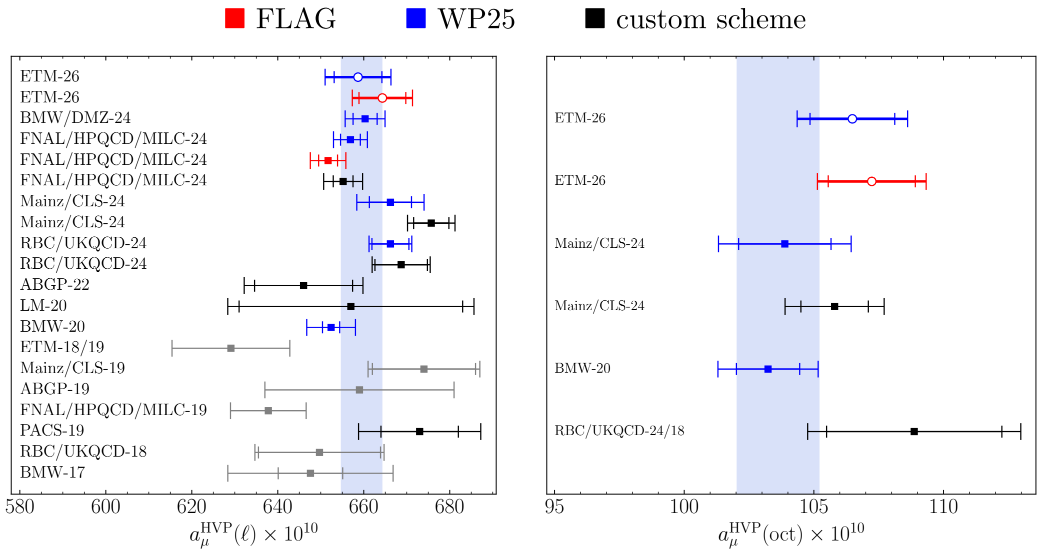
Summary and conclusion

$$a_\mu^{\mathrm{HVP,FLAG}}(I{=}1) = 597.9(4.9)_{\mathrm{stat}}(2.9)_{\mathrm{syst}}(2.7)_L[6.3] \times 10^{-10}$$
$$a_\mu^{\mathrm{HVP,WP25}}(I{=}1) = 592.8(5.0)_{\mathrm{stat}}(3.9)_{\mathrm{syst}}(2.7)_L[6.9] \times 10^{-10}$$

$$a_\mu^{\mathrm{HVP,FLAG}}(\mathrm{oct}) = 107.2\,(1.7)_{\mathrm{stat}}\,(1.3)_{\mathrm{syst}}\,[2.1] \times 10^{-10}$$
$$a_\mu^{\mathrm{HVP,WP25}}(\mathrm{oct}) = 106.5\,(1.6)_{\mathrm{stat}}\,(1.4)_{\mathrm{syst}}\,[2.1] \times 10^{-10}$$

- $\sim 1\%$ precision for $I = 1$
- $\sim 2\%$ precision for $I = 0$ octet
- Finalizing results: $a_\mu^{\mathrm{HVP}}(c)$, $a_\mu^{\mathrm{HVP}}(s)$, window observables
- Outlook: first-principles isospin-breaking and electromagnetic corrections

Comparison with other collaborations



Thank You!

Backup

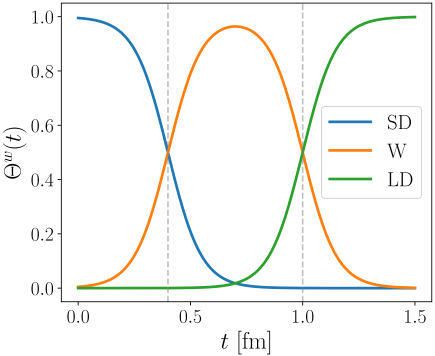
Windows

$$a_\mu^{\mathrm{HVP},w} = 2\alpha_{\mathrm{em}}^2 \int_0^\infty dt\, t^2 K(m_\mu t)\, \Theta^w(t)\, V(t)$$

$$\Theta^{\mathrm{SD}}(t) = 1 - \frac{1}{1+e^{-2(t-t_1)/\Delta}}$$

$$\Theta^{\mathrm{W}}(t) = \frac{1}{1+e^{-2(t-t_1)/\Delta}} - \frac{1}{1+e^{-2(t-t_2)/\Delta}}$$

$$\Theta^{\mathrm{LD}}(t) = \frac{1}{1+e^{-2(t-t_2)/\Delta}}$$



$$t_1 = 0.4 \text{ fm}, \quad t_2 = 1.0 \text{ fm}, \quad \Delta = 0.15 \text{ fm}$$

Scale setting with w_0 and f_π

Gradient flow

$$\frac{d}{dt}B_\mu(x,t) = -\frac{\delta S[B]}{\delta B_\mu(x,t)}\,, \quad B_\mu(x,0) = A_\mu(x)$$

The flow smooths the UV fluctuations

Flowed energy density and w_0 [Luscher and Peter Weisz, 2011; Luscher, 2013]

$$E(t) = \frac{1}{4}\left\langle G^a_{\mu\nu}(t) G^a_{\mu\nu}(t)\right\rangle \qquad W(t) = t\frac{d}{dt}[t^2 E(t)]$$

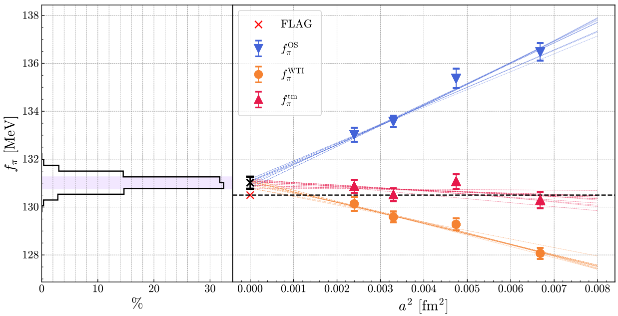
$$\boxed{W(t = w_0^2) = 0.3}$$

WP25 scheme

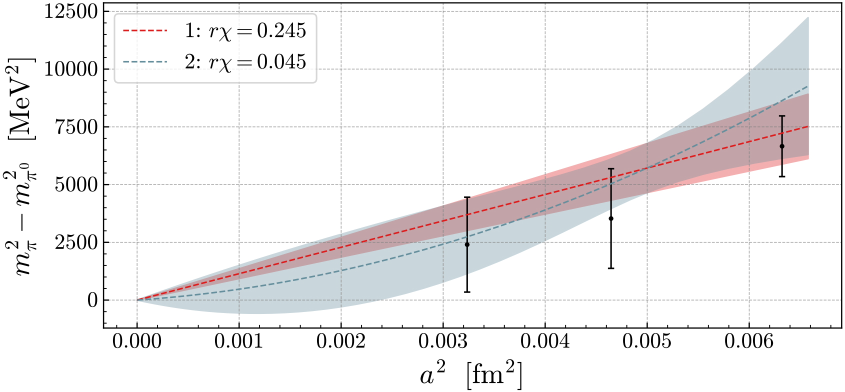
- WP25 iso QCD definition
  $P_\pi^{\mathrm{WP25}} = [m_\pi^2 w_0^2]^{\mathrm{WP25}}, \quad P_K^{\mathrm{WP25}} = [m_K^2 w_0^2]^{\mathrm{WP25}}, \quad P_{D_s}^{\mathrm{WP25}} = [m_{D_s} w_0]^{\mathrm{WP25}}$
- $O(a^2)$ modification to reduce charm mistuning $P_{D_s}^{\mathrm{WP25}'} = [m_{D_s} w_0]^{\mathrm{WP25}} + Ca^2$
- Solve

$$\vec{P}^{\mathrm{sim}} + (m_0 - m_0^{\mathrm{sim}}) \frac{d\vec{P}}{dm_0}\bigg|_{\mathrm{sim}} + \sum_{f=\ell,s,c} (m_f - m_f^{\mathrm{sim}}) \frac{d\vec{P}}{dm_f}\bigg|_{\mathrm{sim}} = \vec{P}^{\mathrm{WP25}'} ,$$

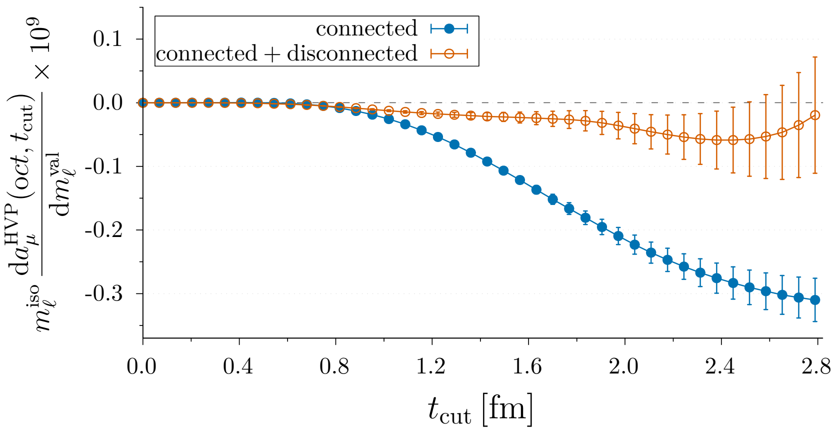
- Compute f_π from conserved current and local current tm and OS



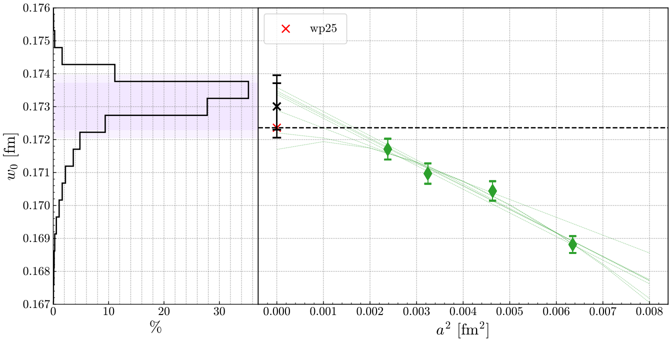
Pion mass splitting



Light quark mass dependence

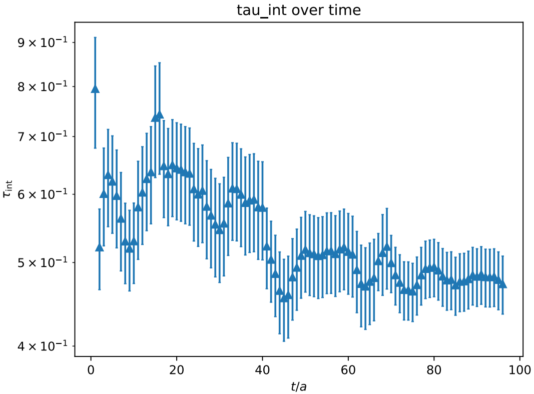


w_0 extrapolation in a^2



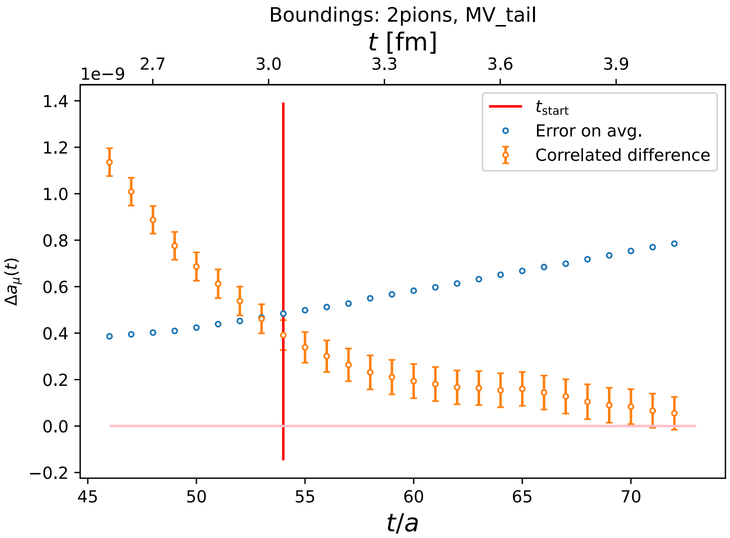
Bootstrap & binning

- Γ-method for autocorrelation [Wolff, 2004]
- Binning: $\tau_{\text{int}} < 0.5$
- $N_b = 500$ bootstrap samples
- Trapezoidal TMR integration, bootstrap-by-bootstrap

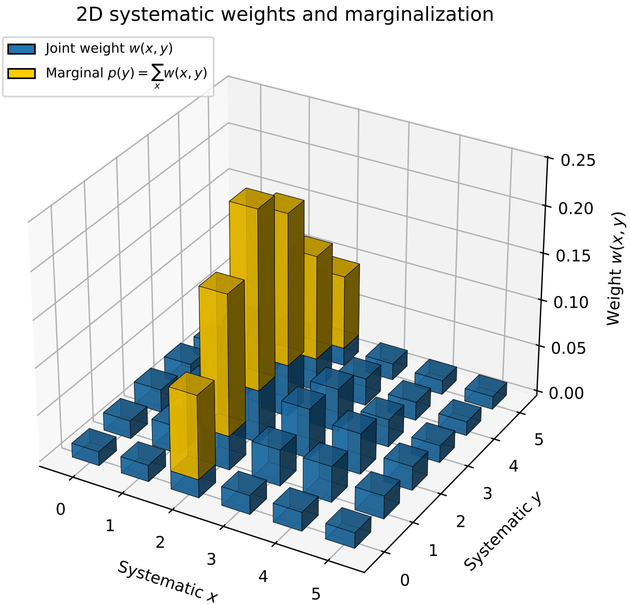


$\tau_{\text{int}}(t)$ for $V(t)$ on D96 (TM)

Bounding fit example: D96, TM, moderate



Breaking down the different systematic effects



Key idea

- Blue: joint weights $w(x, y)$.
- Yellow: marginal $p(y) = \sum_x w(x, y)$.

Goal

- Repeat for every systematic.
- Build the error budget.

Error budget table

Law of total variance

$$\sigma^2_{\text{tot}}\Big|_k = \underbrace{\mathbb{E}[\text{Var}(a_\mu|k)]}_{\text{stat. + other syst.}} + \underbrace{\text{Var}(\mathbb{E}[a_\mu|k])}_{\text{syst. from } k} = \sum_m w_m \sigma_m^2 + \sum_m w_m (a_{\mu,m} - \bar{a}_{\mu,k})^2$$

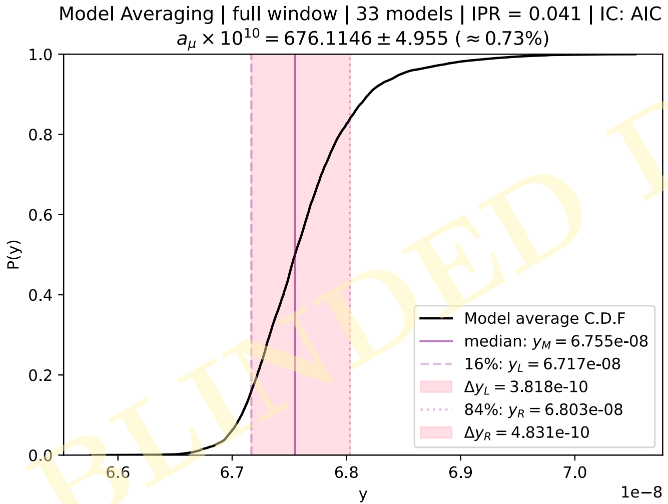
- **Estimator of the mean** (AIC-weighted):

$$\mathbb{E}[a_\mu|k] = \sum_{m \in \mathcal{M}_k} w_m a_{\mu,m}$$

Total systematic and statistical errors

We compute the total error and its

Bern analysis: error budget plot (example, blinded)



Error budget	$\Delta a_\mu \cdot 10^{10}$
tot	4.9550
stat	4.6139
syst_tot	1.8067
n_ens	0.0000
t_thr	0.1052
t_cut	0.1317
dt_plateau	0.1207
extrapolation	1.7499

History of ETMC simulations

Years	N_f	m_π	Reference(s)
2004 - 2010	2	> 200 MeV	[Jansen and Urbach, 2006]
2010 - 2014	$2+1+1$	> 200 MeV	[Carrasco et al., 2014]
2014 - 2018	2	$= 135$ MeV	[Abdel-Rehim et al., 2017]
2018 - 2024	$2+1+1$	$= 135$ MeV	[Alexandrou et al., 2021] [Alexandrou et al., 2023] [Alexandrou et al., 2024]

Remarks on fermions (ideal situation)

What we would like to have:

If we knew the bare masses exactly at the isoQCD physical point, we would like to simulate the probability distribution:

$$P[U] = e^{-S_{\text{YM}}} \prod_f \mathcal{D}_f^{\text{OS}}(a\mu_f^{\text{phys}}, am_{0,f}^{\text{phys}})$$

- $\mathcal{D} = \det D$: determinant of Dirac operator
- OS: Osterwalder-Seiler fermions (or any other type of fermions)

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Problem

- We don't know a priori the exact values of the bare parameters

Remarks on fermions (practice)

What we actually have:

Despite being close to the isoQCD physical point, our probability measure is still:

$$P[U] = e^{-S_{\mathrm{YM}}} \prod_f \det D_f^{\mathrm{TM}}(a\mu_f^{\mathrm{sim}}, am_{0,f}^{\mathrm{sim}})$$

TM: Twisted Mass fermions [Frezzotti, Grassi, et al., 2001; Frezzotti and Rossi, 2004a; Shindler, 2008]

- Strictly positive fermionic determinant for $\mu \neq 0$
- If $am_0 = am_{cr} \to$ *maximal twist* $\to O(a)$ improvement

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However, we need to correct the mistuning:

- $a\mu_f^{\mathrm{sim}} \neq a\mu_f^{\mathrm{phys}}$
- $am_{0,f}^{\mathrm{sim}} \neq am_{cr,f}$

Mixed Action setup [Frezzotti and Rossi, 2004b]

Total action

All this is equivalent to a mixed action approach:

$$S = S_{\mathrm{YM}} + \sum_f (S_F)^{(f)}_{\mathrm{TM}} + S^{(f)}_{\mathrm{ghost}} + (S_F)^{(f),val}_{\mathrm{OS}}$$

For the gauge part we use the mean-field improved Iwasaki action [Iwasaki, 1985; S. Aoki, Frezzotti, and P. Weisz, 1999]:

$$S_{\mathrm{YM}} = \frac{\beta}{N_c} \sum_x \sum_{\mu<\nu} b_0 \left[1 - \mathrm{Re}\,\mathrm{Tr}\, U^{1\times 1}_{\mu\nu}(x)\right] + b_1 \left[1 - \mathrm{Re}\,\mathrm{Tr}\, U^{1\times 2}_{\mu\nu}(x)\right].$$

Fermions (sea contributions)

The Dirac operator for the sea quarks is the TM one:

- Light quarks (isosymmetric limit: $m_u = m_d = m_\ell$):

$$D_\ell^{\mathrm{TM}} = \gamma_\mu \nabla_\mu[U] - i\tau^3\gamma_5 \left(W^{cl}[U] + m_0\right) + \mu_\ell$$

- Heavy quarks:

$$D_h^{\mathrm{TM}} = \gamma_\mu \nabla_\mu[U] - i\tau^1\gamma_5 \left(W^{cl}[U] + m_0\right) + \mu_\sigma + \tau^3\mu_\delta$$

where $W^{cl}[U]$ is the the Wilson-Clover term [Sheikholeslami and Wohlert, 1985].

The action reads:

$$(S_F)_{\mathrm{TM}} = \sum_x \bar\Psi_\ell\, D_\ell^{\mathrm{TM}}\, \bar\Psi_\ell + \bar\Psi_h\, D_h^{\mathrm{TM}}\, \Psi_h$$

Fermions (sea and ghost pseudo-fermions)

The correction of the mistunings comes from both the sea and the valence.

OS fermions

The observables and the mistuning corrections are computed with OS fermions

$$D^f_{\mathrm{OS}} = \gamma_\mu \bar{\nabla}_\mu[U] - i r_f \gamma_5 \left(W^{\mathrm{cl}}[U] + m_0 \right) + m_f$$

where $r_{u,c} = 1$ and $r_{d,s} = -1$.

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Applying the corrections in the sea is the same as:

- Including ghost pseudo-fermions through $S^{(f)}_{\mathrm{ghost}} \rightarrow$ cancel simulated fermionic determinant
- Correct determinants from OS fermions at the isoQCD physical point and critical mass

Summary on Fermions

Corrections found with OS fermions $\rightarrow$ probability measure is:

$$\boxed{P[U] = \underbrace{e^{-S_{\rm YM}(g_0)}}_{\text{YM action}} \times \underbrace{\mathcal{D}^{\ell}_{\rm TM}(\mu_\ell^{\rm sim}, m_{\rm cr}^{\rm sim})\mathcal{D}^{h}_{\rm TM}(\mu_\sigma^{\rm sim}, \mu_\delta^{\rm sim}, m_{\rm cr}^{\rm sim})}_{\text{TM det. from sim}} \times \underbrace{\prod_f \frac{\mathcal{D}^f_{\rm OS}(m_f, m_{\rm cr})}{\mathcal{D}^f_{\rm OS}(m_f^{\rm sim}, m_{\rm cr}^{\rm sim})}}_{\text{Correction factor}}}$$

Matching TM and OS fermions [Frezzotti and Rossi, 2004b]

$$m_\ell = \mu_\ell, \quad m_s = \mu_\sigma - \frac{Z_P}{Z_S}\mu_\delta, \quad m_c = \mu_\sigma + \frac{Z_P}{Z_S}\mu_\delta$$

- Light quarks: $\mathcal{D}_{\rm TM} = \mathcal{D}_{OS}$
- Heavy quarks: $\frac{\mathcal{D}^h_{TM}(\mu_\sigma^{\rm sim}, \mu_\delta^{\rm sim}, m_0^{\rm sim})}{\mathcal{D}^s_{OS}(m_s^{\rm sim}, m_0^{\rm sim})\cdot\mathcal{D}^c_{OS}(m_c^{\rm sim}, m_0^{\rm sim})} = 1 + O(a^2)$

Wrap-up on the mixed-action setup

(Reminder) What we want

$$P[U] = e^{-S_{YM}(g_0)} \times \prod_f \mathcal{D}^f_{OS}(m_f, m_0)$$

What we have

$$P[U] = e^{-S_{YM}(g_0)} \times \prod_f \mathcal{D}^f_{OS}(m_f, m_0) \times \left(1 + O(a^2)\right)$$

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What does it mean in practice?

1. Ensemble generation: good guesses for bare parameters
2. Find the slopes of observables with respect to sea and valence
3. Correct the bare parameters to reach the isoQCD physical point.

Diagrammatic representation of sea effects

Correction factor for sea effects

Sea quarks mistuning brings the following correction factor:

$$W_f(m_f, m_{cr}) = \hat{W}_f(m_{cr})\bar{W}_f(m_f)$$

$$\hat{W}_f(m_{cr}) = \frac{D^f_{OS}(m_0^{\text{sim}}, m_{cr})}{D^f_{OS}(m_0^{\text{sim}}, m_0^{\text{sim}})}, \quad \bar{W}_f(m_f) = \frac{D^f_{OS}(m_f, m_{cr})}{D^f_{OS}(m_f^{\text{sim}}, m_{cr})}$$

Expansion parameters

$$\Delta m_f = m_f - m_f^{\text{sim}}, \quad \Delta m_{\text{cr}} = m_{\text{cr}} - m_0^{\text{sim}}$$

They are very small (both $\sim O(1\%)$) $\rightarrow$ we treat them as being of equal order Δm.

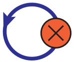
Diagrammatic representation of sea effects

Jacobi formula

For a parameter-dependent matrix $A(\lambda)$:

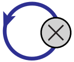
$$\partial_\lambda \det(A) = \det(A) \cdot \mathrm{Tr}\left(A^{-1}\partial_\lambda A\right)$$

$$\hat{W}_f(m_{cr}) = 1 + \Delta m_{\rm cr}\, \mathrm{Tr}\left[(-ir_f\gamma_5)\frac{1}{D_{OS}^{\rm sim}}\right] + O(\Delta m)^2 = 1 + \Delta m_{\rm cr} \cdot$$



$$+ O(\Delta m)^2$$

$$\bar{W}_f(m_f) = 1 + \Delta m_f\, \mathrm{Tr}\left[\frac{1}{D_{OS}^{\rm sim}}\right] + O(\Delta m)^2 = 1 + \Delta m_f \cdot$$



$$+ O(\Delta m)^2$$

Sea effects as a 1st order perturbation

$$S^{\text{phys}} = S^{\text{sim}} + \Delta S$$

$$\langle O \rangle^{\text{phys}} = \frac{\int \mathcal{D}U \, O[U] \, e^{-S^{\text{phys}}}}{\int \mathcal{D}U \; e^{-S^{\text{phys}}}} \approx \langle O \rangle^{\text{sim}} + \langle O \Delta S \rangle^{\text{sim}} - \langle O \rangle^{\text{sim}} \langle \Delta S \rangle^{\text{sim}}$$

Physical and critical mass corrections

$$\partial^{\text{sea}}_{\text{cr},f} O[U] = \langle O[U] \quad \rangle^{\text{sim}} - \langle O[U] \rangle^{\text{sim}} \langle \quad \rangle^{\text{sim}}$$

$$\partial^{\text{sea}}_{f} O[U] = \langle O[U] \quad \rangle^{\text{sim}} - \langle O[U] \rangle^{\text{sim}} \langle \quad \rangle^{\text{sim}}$$

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