Mainz update on the HLbL

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- Update on the determination from the pion transition form factor
- New ideas for future calculations
  - Factorizing the HLbL kernel
  - The electroweak triangle diagram

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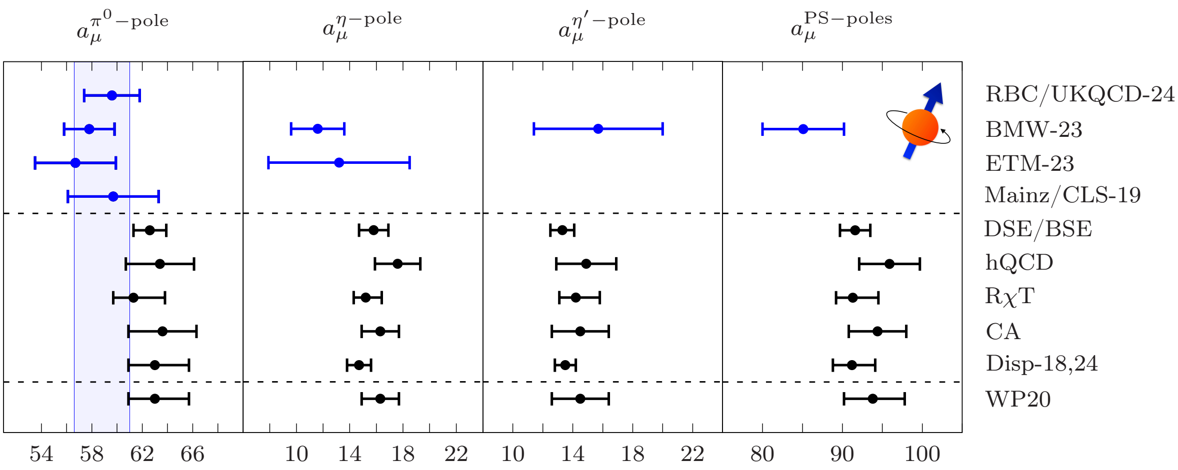
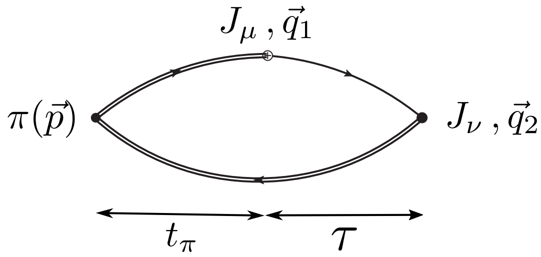


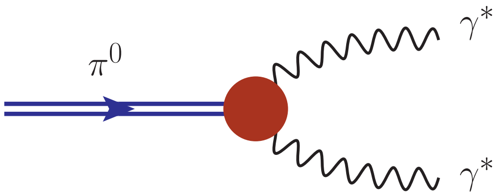
- White paper '25 consensus :

$$a_\mu^{\pi^0\text{-pole}} = a_\mu^{\pi^0\text{-pole,conn}} \pm \left| a_\mu^{\pi^0\text{-pole,disc}} \right| = 58.8(1.1)_{\text{stat}}(1.1)_{\text{syst}}(1.6)_{\text{disc}}[2.2]_{\text{tot}} \times 10^{-11}$$

→ not all collaborations agree on the sign of the quark-disconnected contribution

→ disconnected contribution $\sim 1\%$ at the level of the TFF





- We start with the following amplitude (on-shell pion)

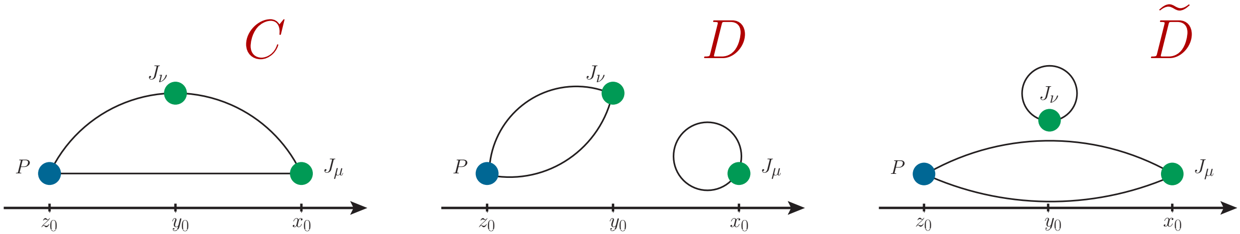
$$\widetilde{A}^{(sv)}_{\mu\nu}(\tau,\vec{q}_1,\vec{p}) = \lim_{z_0\to-\infty} e^{E_{\pi,\vec{p}}(y_0-z_0)}\,\mathcal{C}^{(3,sv)}_{\mu\nu}(y_0+\tau,y_0,z_0;\vec{q}_1,\vec{p})$$

$$\widetilde{A}^{(vs)}_{\mu\nu}(\tau,\vec{q}_1,\vec{p}) = e^{-E_{\pi,\vec{p}}\tau}\ \widetilde{A}^{(sv)}_{\nu\mu}(-\tau,\vec{q}_2,\vec{p})$$

- Relation with the form factor

$$\epsilon_{\mu\nu\alpha\beta}\,q_1^\alpha q_2^\beta\,\mathcal{F}_{sv}(q_1^2,q_2^2) = i^{n_0}\frac{2E_{\pi,\vec{p}}}{Z_\pi}\int_{-\infty}^{\infty} d\tau\, e^{\omega_1\tau}\,\widetilde{A}^{(sv)}_{\mu\nu}(\tau,\vec{q}_1,\vec{p})$$

$\rightarrow$ $q_1 = (\omega_1,\vec{q}_1)$ and $q_2 = (E_{\pi,\vec{p}}-\omega_1,\vec{p}-\vec{q}_1)$
$\rightarrow$ ω_1 : continuous parameter



$$\mathcal{C}^{(3)}_{\mu\nu}(x_0, y_0, z_0; \vec{q}_1, \vec{p}) \equiv a^6 \sum_{\vec{x},\vec{z}} e^{-i\vec{q}_1(\vec{x}-\vec{y})+i\vec{p}\cdot(\vec{z}-\vec{y})} \langle\, j^{\mathcal{Q}}_\mu(x) j^{\mathcal{Q}}_\nu(y) P^\dagger(z)\,\rangle\,.$$

Wick contractions (e.m. current : $\mathcal{Q}^{\text{e.m.}} = \frac{\lambda^3}{2} + \frac{1}{\sqrt{3}}\frac{\lambda^8}{2}$) :

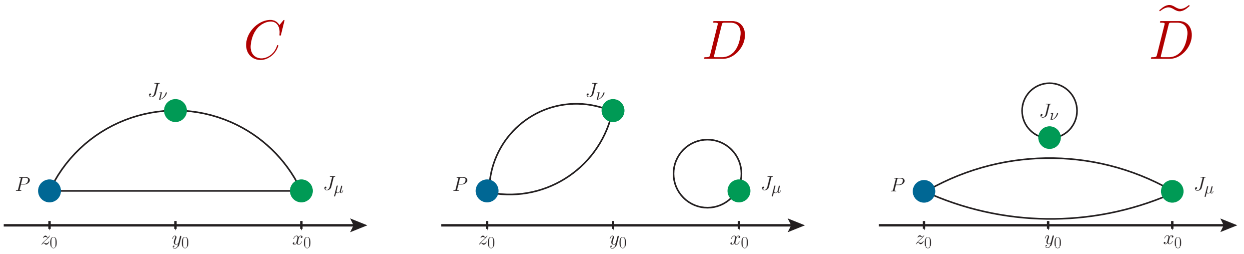
$$\boxed{\mathcal{C}^{(3)}_{\mu\nu} = -\frac{1}{3\sqrt{2}}\left(2\, C_{\mu\nu} + D^{(l-s)}_{\mu\nu} + \widetilde{D}^{(l-s)}_{\mu\nu}\right).}$$

In iso-symmetric QCD : non-vanishing contribution require one isovector and one isoscalar current

$$\mathcal{C}^{(3)}_{\mu\nu} = \mathcal{C}^{(3,sv)}_{\mu\nu} + \mathcal{C}^{(3,vs)}_{\mu\nu} \qquad \text{with} \qquad \begin{aligned} \mathcal{C}^{(3,sv)}_{\mu\nu} &= -\frac{1}{3\sqrt{2}}\left(C_{\mu\nu} + D^{(l-s)}_{\mu\nu}\right) \\ \mathcal{C}^{(3,vs)}_{\mu\nu} &= -\frac{1}{3\sqrt{2}}\left(C_{\mu\nu} + \widetilde{D}^{(l-s)}_{\mu\nu}\right) \end{aligned}$$

In particular, the following combination of disconnected diagrams has an interpretation within QCD

$$\boxed{\mathcal{C}^{(3,sv)}_{\mu\nu} - \mathcal{C}^{(3,vs)}_{\mu\nu} = -\frac{1}{3\sqrt{2}}\left[D^{(l-s)}_{\mu\nu} - \widetilde{D}^{(l-s)}_{\mu\nu}\right]}$$



In the kinetic regime $x_0 > y_0 > z_0$:

$$\mathcal{C}^{(3,sv)}_{\mu\nu} = -\frac{1}{3\sqrt{2}}\left(C_{\mu\nu} + D^{(l-s)}_{\mu\nu}\right)$$

→ Long-distance $\pi\pi$-contribution between y_0 and x_0 must cancel between C and D

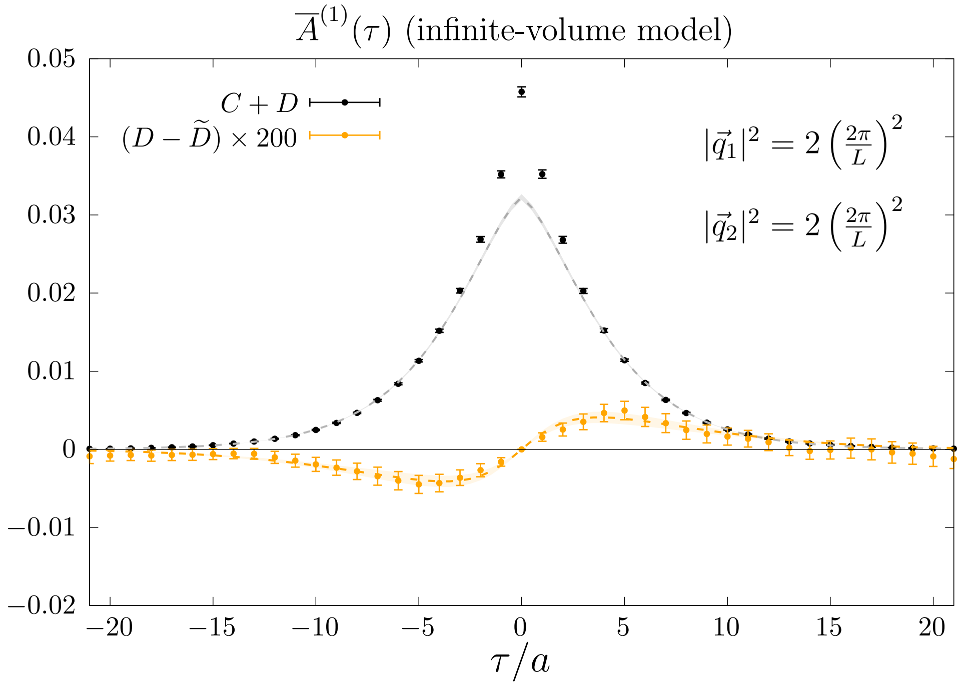
→ At long-distance C and D must have opposite signs

$$\mathcal{C}^{(3,vs)}_{\mu\nu} = -\frac{1}{3\sqrt{2}}\left(C_{\mu\nu} + \widetilde{D}^{(l-s)}_{\mu\nu}\right)$$

→ Isoscalar contribution between y_0 and x_0 must cancel between C and $\widetilde{D}$

$$\mathcal{C}^{(3)}_{\mu\nu} = -\frac{1}{3\sqrt{2}}\left(2\,C_{\mu\nu} + D^{(l-s)}_{\mu\nu} + \widetilde{D}^{(l-s)}_{\mu\nu}\right)$$

$$\mathcal{C}^{(3,sv)}_{\mu\nu} - \mathcal{C}^{(3,vs)}_{\mu\nu} = -\frac{1}{3\sqrt{2}}\left[D^{(l-s)}_{\mu\nu} - \widetilde{D}^{(l-s)}_{\mu\nu}\right]$$



- The disconnected contribution is small [Mainz '19]
- The asymmetry $\mathcal{C}^{(3,sv)}_{\mu\nu} - \mathcal{C}^{(3,vs)}_{\mu\nu}$ is further reduced

  $\rightarrow$ arises from the fact that the isovector current couples to $\pi^+\pi^-$ states, while the isoscalar only couples to $\pi^+\pi^-\pi^0$ states

- The 2019 analysis focused on the connected contribution : the disconnected contribution was treated as a small correction
- New analysis in the isospin basis
  $\rightarrow$ possible thanks to a much better statistical precision for the disconnected contribution
  $\rightarrow$ gives access to the physical observable $\mathcal{C}_{\mu\nu}^{(3,sv)} - \mathcal{C}_{\mu\nu}^{(3,vs)}$
- Simple VMD or LMD+V models necessarily imply $\mathcal{C}_{\mu\nu}^{(3,sv)} - \mathcal{C}_{\mu\nu}^{(3,vs)} = 0$ (note that in 2019, the lattice data were described by a z-expansion).
- Use a model, inspired from dispersive analyses, to describe both the total contribution to the form factor **and** the asymmetry

Starting point : dispersive formula, inspired by [Hoferichter et. al '14]

$$\mathcal{F}_{vs}(q_1^2, q_2^2) = \frac{1}{12\pi^2} \int_{4M_\pi^2}^{\infty} ds \frac{q_\pi(s)^3 \, F_\pi^V(s)^* f_1(s, q_2^2)}{s^{1/2}\,(s - q_1^2)}.$$

- $F_\pi^V(s)$ described by a Gounaris-Sakurai model
- $f_1(s, q_2^2)$: amplitude for the process $\gamma_s^* \to \pi^+\pi^-\pi^0$.

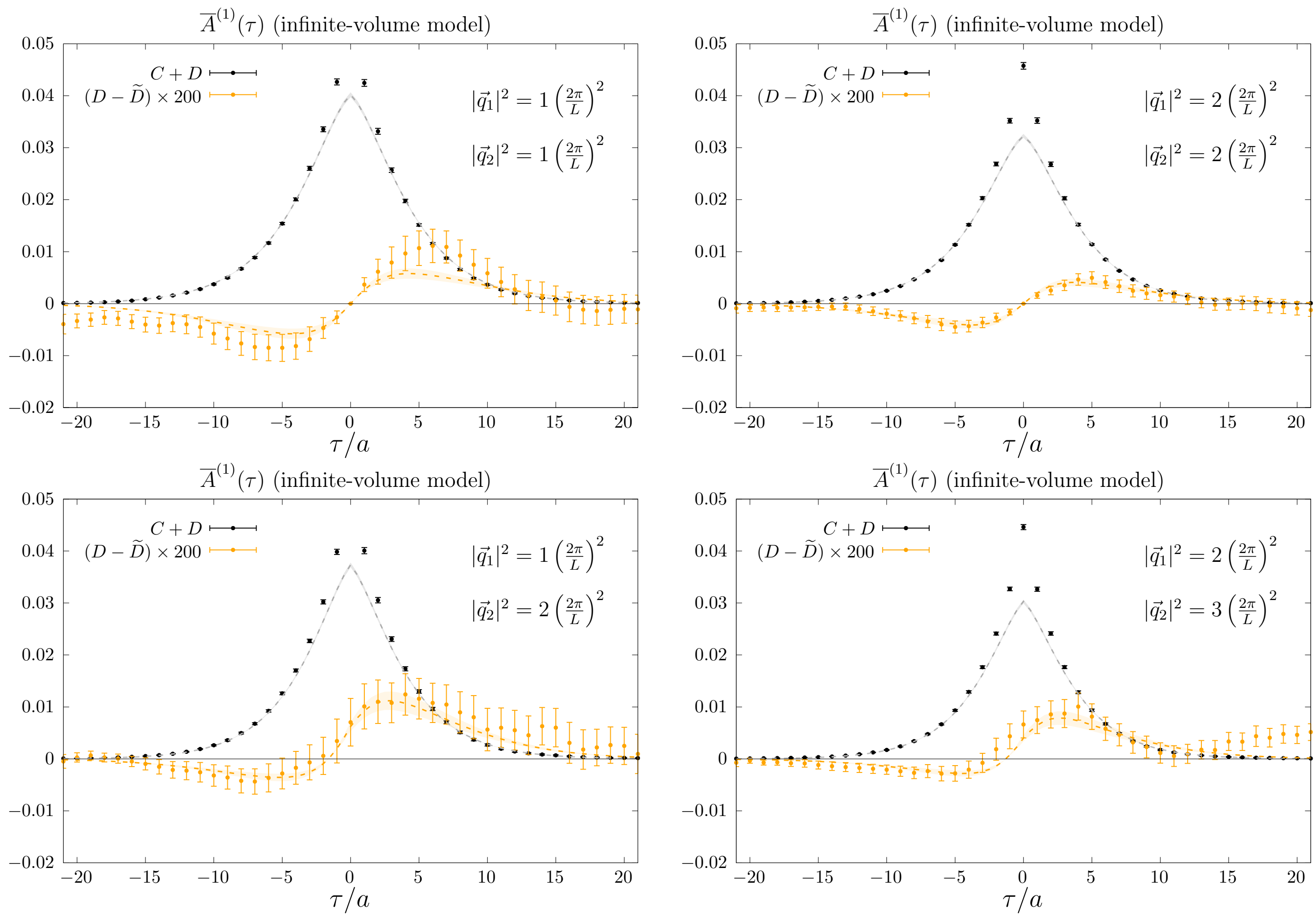
  $\to$ space-like : simple parametrization for f_1, with two resonances ($M_{V_1} = 0.742$ MeV and $M_{V_2} = 1.4$ GeV)

  $\to$ start with only two fit parameters
- Translate the model into the time-momentum representation $\mathcal{F} \to \widetilde{A}_{\mu\nu}(\tau)$
- This model is expected to provide a good description of the data for sufficiently large τ

  $\to$ the model cannot describe the cusp at $\tau = 0$

  $\to$ currently, we simply exclude connected data with $|\tau|/a < 5$

  $\to$ but the model can be easily extended to describe this short-distance contribution



- Good description of the asymmetry

- Update on the determination from the pion transition form factor
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Isospin-violation part of the HVP with Pauli-Villars regulated photon propagator :

$$a_\mu^{\mathrm{HVP},38} = -\frac{e^2}{2}\int_{x,y,z} H_{\lambda\sigma}(z)\delta_{\mu\nu}[\mathcal{G}(x-y)]_\Lambda \left\langle j^3_\lambda(z) j^{em}_\mu(x) j^{em}_\nu(y) j^8_\sigma(0)\right\rangle + C_T(\Lambda),$$

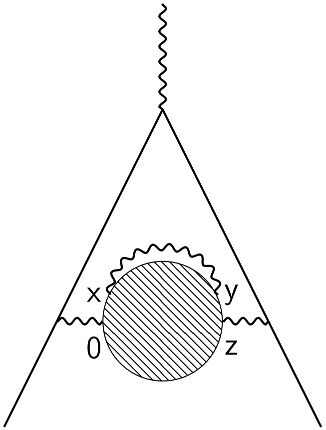
$$[\mathcal{G}(x)]_\Lambda = \frac{1}{4\pi^2|x|^2} - 2G^{(1)}_{\frac{\Lambda}{\sqrt{2}}}(x) + G^{(1)}_\Lambda(x),$$

$$G^{(1)}_m(x) = \frac{mK_1\left(m|x|\right)}{4\pi^2|x|}.$$

- Photon propagator depends on difference between x and y vertex

  $\rightarrow$ Volume square sum
- Can be avoided by setting y to single point

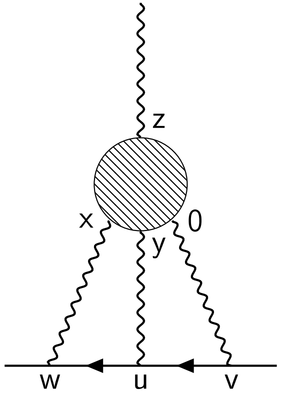
  or by factorizing $[\mathcal{G}(x-y)]_\Lambda$
- In 2603.13086 we recently tested two possible factorizations



$$a_\mu^{\mathrm{HLbL}} = \frac{m_\mu e^6}{3} \int d^4x\, d^4y \overbrace{\left(- \int d^4z\, z_\rho \left\langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0) \right\rangle_{\mathrm{QCD}} \right)}^{i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)} \times$$

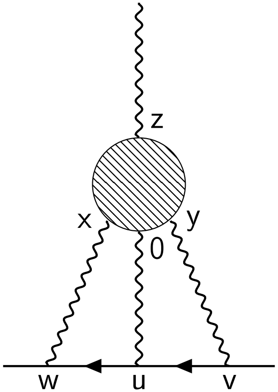
$$\times \overbrace{\left(\frac{1}{16 m_\mu^2} \int d^4u\, d^4v\, d^4w\, G_0(w-x) G_0(u-y) G_0(v) e^{-ip\cdot(w-v)} \times \right.}^{\mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(p,x,y)}$$

$$\left. \times \operatorname{Tr}\left\{ [\gamma_\rho \gamma_\sigma](-i\not{p} + m_\mu)\gamma_\mu S(w-u)\gamma_\nu S(u-v)\gamma_\lambda(-i\not{p}+m_\mu) \right\} \right)$$



- Calculate $i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)$ with lattice QCD
- Sum over x and z explicitly, reduce y integral to 1D integral over $|y|$
- Use precalculated $\mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(p,x,y)$

$$a_\mu^{\mathrm{HLbL}} = \frac{m_\mu e^6}{3} \int d^4u\, G(u) \int d^4x\, d^4y \left(- \int d^4z\, z_\rho \left\langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0) \right\rangle_{\mathrm{QCD}} \right) \times$$
$$\times \left(\frac{1}{16 m_\mu^2} \mathrm{Tr} \left\{ [\gamma_\rho \gamma_\sigma](-i\not{p} + m_\mu) \gamma_\mu L^f(p, x-u) \gamma_\nu L^f(p, u-y) \gamma_\lambda (-i\not{p} + m_\mu) \right\} \right)$$
$$L^f(p,x) = \int d^4w\, G_0(x-w) e^{-ip\cdot w} S(w)$$



- For fixed u, we can now sum over x, y and z using sequential propagators
- Averaging technique for $\hat{p}$ not applicable

  $\rightarrow$ Last integral over u can be reduced to 2D (not 1D)
- No test calculation from Mainz performed yet

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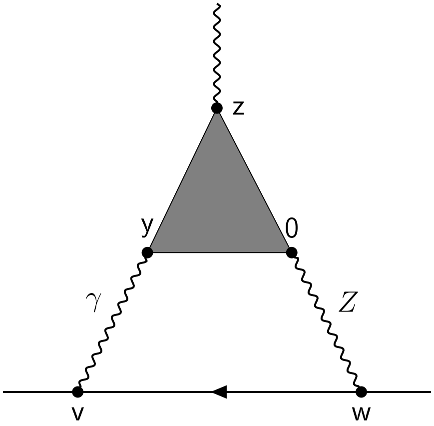
Similar structure and methods as HLbL

- Involves all three gauge interactions of the standard model
- Triangle : Must take into account both leptons and quarks
- The latter lead to non-perturbative contributions
- No lattice calculations yet
- Contribution of all three fermion families :

  $a_\mu^\triangle = (-14.3 \pm 0.3) \times 10^{-11}$ (Phys. Rev. Lett. 134.20 (2025))
- Commensurate to precision of current experimental world average

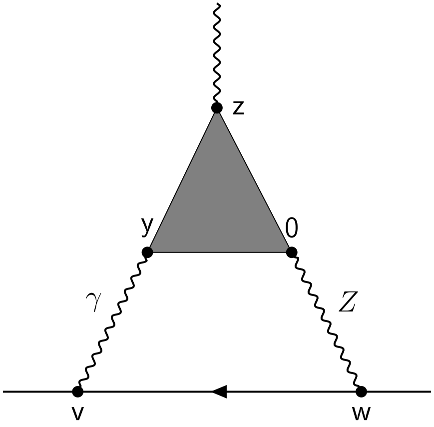
  $\delta a_\mu = 14.5 \times 10^{-11}$



Similar structure and methods as HLbL

$$a_\mu^{\triangle} = \frac{16 m_\mu e^2 G_F}{3\sqrt{2}} \int d^4y \, \mathcal{H}_{[\rho,\sigma];\nu\lambda}(y) \times$$
$$\times \int d^4z \, z_\rho \langle V_\nu(y) V_\sigma(z) A_\lambda(0) \rangle_{\text{QCD+leptons}}$$

- Calculate $\langle V_\nu(y) V_\sigma(z) A_\lambda(0) \rangle_{\text{QCD+leptons}}$ with lattice QCD
  - Possible for first two fermion generations
  - Lepton : Same lattice spacing, larger or infinite volume
- Use precalculated $\mathcal{H}_{[\rho,\sigma];\nu\lambda}(y)$



$$
\begin{aligned}
\mathcal{H}_{[\rho,\sigma];\nu\lambda}(y) = &\frac{1}{16m_\mu^2}\left(M_Z^2\delta_{\alpha\lambda} - \partial_\alpha^{(y)}\partial_\lambda^{(y)}\right)\times \\
&\times \left\langle \mathrm{Tr}\left\{[\gamma_\rho,\gamma_\sigma](-i\not{p}+m_\mu)\gamma_\nu \mathcal{L}^{(1,f)}_{m_\mu,M_Z,0}(p,p,0,y)\gamma_\alpha^Z(-i\not{p}+m_\mu)\right\}\right\rangle_{\hat{p}}, \\
\mathcal{L}^{(1,f)}_{m,M_1,M_2}(p,p',x,y) = &\int d^4z \int d^4v\, e^{i(p\cdot z - p'\cdot v)} G_{M_2}(y-v) S_m(v-z) G_{M_1}(x-z), \\
p = im_\mu\hat{p}, &\quad \gamma_\alpha^Z = \gamma_\alpha\left(\left(-\frac{1}{4}+s_W^2\right)+\frac{1}{4}\gamma_5\right)
\end{aligned}
$$

- Analytical methods available in JHEP06(2026)255
- Function $\mathcal{L}^{(1,f)}_{m,M_1,M_2}(p,p',x,y)$ known in Feynman parameter representation
- Tested and cross-checked in the continuum
- No test runs from Mainz yet

Conclusion and outlook

HLbL from pion TFF :

- Continued disagreement on the relative sign of the disconnected contribution between collaborations
- Isospin decomposition can make comparison with phenomenology easier

Future HLbL plans :

- Test factorization approach for direct lattice calculation of HLbL
- Perform lattice calculation of non-perturbative part of EW triangle diagram