

Towards hadronic light-by-light contribution to the muon $g - 2$ with C^* boundary conditions

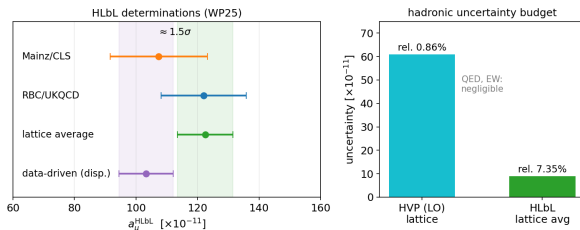
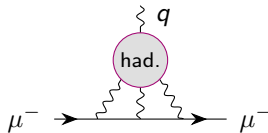
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on behalf of the RC* collaboration

ETH Zürich

Muon $g - 2$ Theory Initiative, 9th Plenary Workshop, UConn, August 2026

An alternate route to HLbL

- a_{μ}^{HLbL} : **hadronic light-by-light scattering** dressing the vertex, $\mathcal{O}(\alpha^3)$ (right).
- As HVP improves toward 124 ppb, the **HLbL uncertainty becomes more significant** (relative uncertainty: 7.4% on HLbL vs. 0.86% on HVP) [WP25].
- Complete lattice determinations: RBC/UKQCD, Mainz/CLS, BMW; ETMC advancing at the physical point; mild $\approx 1.5\sigma$ tension with data-driven.
- This talk: a **new route with independent systematics**.



lattice: RBC/UKQCD, PRL 124 (2020) 132002; PRD 111 (2025) 014501
Mainz/CLS, EPJ C 81 (2021) 651; 82 (2022) 664 · ETMC, 2512.21155 · BMW, 2411.11719

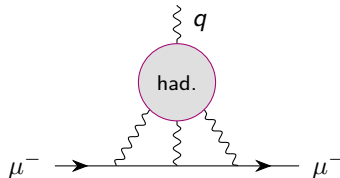
data-driven: Hoferichter, Stoffer, Zillinger, PRL 134 (2025) 061902 · WP25, 2505.21476

The four-current correlator

Quantity of interest: the muon vertex dressed by the hadronic four-current correlator,

$$\Pi_{\mu\nu\lambda\rho}(q_1, q_2, q_3) = \int_{x_1 x_2 x_3} e^{-i(q_1 x_1 + q_2 x_2 + q_3 x_3)} \langle J_\mu(x_1) J_\nu(x_2) J_\lambda(x_3) J_\rho(0) \rangle_{\text{QCD}}.$$

The complete determinations (RBC/UKQCD, Mainz/CLS) compute this **full four-current correlator**, with dedicated variance-reduction machinery for the long-distance signal-to-noise; notably, the moment method computes $F_2(0)$ directly, with no momentum extrapolation.



Blum et al. (RBC/UKQCD), PRL 124 (2020) 132002; PRD 111 (2025) 014501 (QED $_\infty$, physical m_π) · Chao et al. (Mainz/CLS), Eur. Phys. J. C 81 (2021) 651

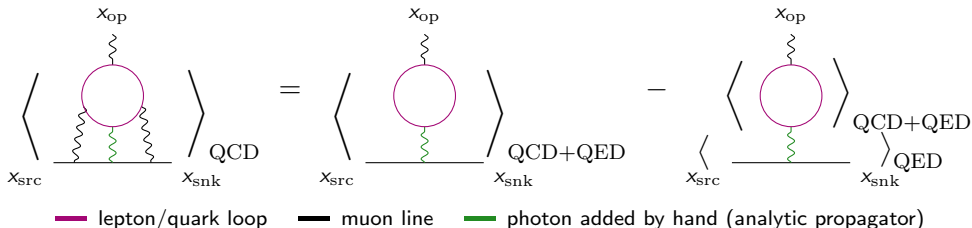
moment method: Blum, Christ, Hayakawa, Izubuchi, Jin, Lehner, PRD 93 (2016) 014503, arXiv:1510.07100

The subtraction method

$$M = \langle AB \rangle_{\text{QCD+QED}} - \langle A \rangle_{\text{QCD+QED}} \langle B \rangle_{\text{QED}}$$

A: loop with two currents (external + internal)

B: muon line with one internal current



- Two two-current correlators on a common ensemble; no four-point function.
- In the difference the lower-order contractions cancel in the path integral; the leading surviving term is the connected HLbL at $\mathcal{O}(\alpha^3)$.

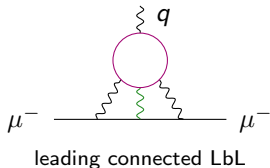
Hayakawa, Blum, Izubuchi, Yamada, PoS LAT2005 (2006) 353, hep-lat/0509016

Blum, Chowdhury, Hayakawa, Izubuchi, PRL 114 (2015) 012001, arXiv:1407.2923

The price and the lever: a controlled cancellation

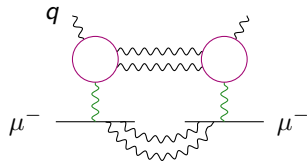
Signal (mean)

Appears only at $\mathcal{O}(\alpha^3)$: $\langle AB \rangle$ and $\langle A \rangle \langle B \rangle$ are each two orders larger in α ; the signal is the survivor of their subtraction.



Noise (variance)

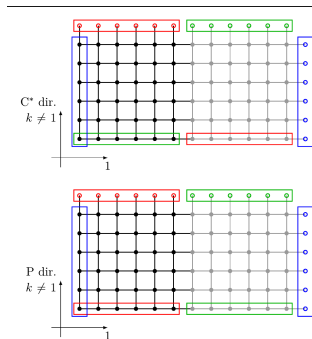
For $\sigma^2 \sim \alpha^n/N$ the signal-to-noise is $\sqrt{N} \alpha^{3-n/2}$; the power n is set by the estimator, not by the observable: noise control is estimator design (quantified near the end).



leading variance topology (LLbL, for the estimator used here)

C* boundary conditions, concisely

- Fields are periodic in space **up to charge conjugation**: $\phi(x + \hat{k}L) = \phi^C(x)$.
- The photon is C-odd, so the **zero mode does not exist by construction**: local, gauge-invariant QED in finite volume, with charged states well defined.
- Implemented in **openQxD** by the **RC*** **collaboration**; dynamical QCD+QED ensembles are available (RC*).



extended (doubled) lattice: physical sites (black), mirror copy (grey)

[openQ*D paper, Fig. 2]

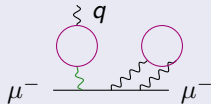
Kronfeld, Wiese, Nucl. Phys. B 357 (1991) 521 · Lucini, Patella, Ramos, Tantalò, JHEP 02 (2016) 076, arXiv:1509.01636

Campos et al. (RC*), Eur. Phys. J. C 80 (2020) 195, arXiv:1908.11673

This project: a three-stage programme

Final objective

Connected + leading disconnected HLbL in fully dynamical QCD+QED (charged sea quarks)



1. QED benchmark (this talk)



exact perturbative target;
periodic boundaries



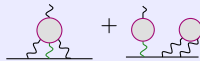
2. QCD + quenched QED quark loop



clover Wilson, three spacings,
 $M_\pi = 420$ MeV



3. dynamical QCD+QED C^* ensembles (RC^*)



leading disconnected automatic;
 RC^* ensembles available

Stage 1 tests method, estimator, and systematics where the exact answer is known.

The leptonic LbL test

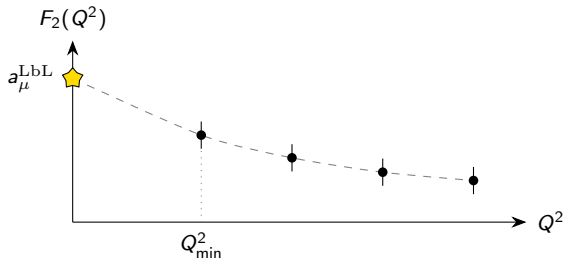
- The quark loop is replaced by a lepton loop: exact benchmark (Laporta-Remiddi)
 $F_2(0)_{\text{LbL}} = 0.371 (\alpha/\pi)^3$ [Laporta, Remiddi, Phys. Lett. B 301 (1993) 440].
- Larger-than-physical coupling $\alpha = 1/4\pi$ ($e = 1$): target $F_2(0) = +6.03 \times 10^{-6}$. Wilson muon, non-compact Feynman-gauge photons; all scales measured on the same ensembles (m_μ, Z_μ, Z_V).
- This talk: coarsest spacing, two geometries:

setup	geometry	N_{cfg}	$\#Q^2$	Q_{min}^2
isotropic	24×8^3	200	4 shells	0.398
anisotropic	$24 \times 16 \times 8^2$	100	10	0.121

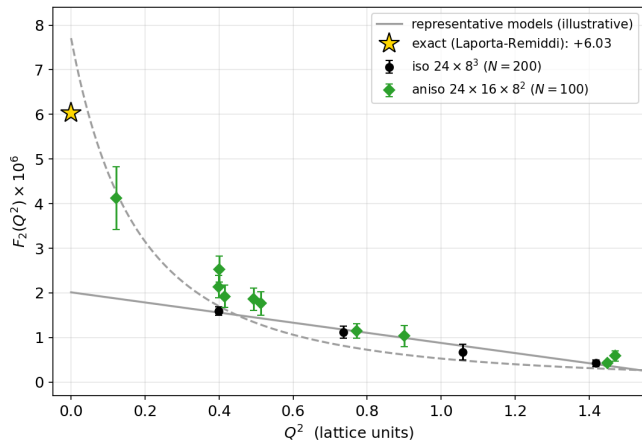
The test targets three questions related to the HLbL: **(a)** implementation and cost, **(b)** the continuum limit, **(c)** the statistical nature of the estimator.

From the lattice correlator to a_μ

- We try to reproduce the static limit $a_\mu^{\text{LbL}} = F_2(0)$, the Pauli form factor of the LbL-dressed vertex at zero momentum transfer.

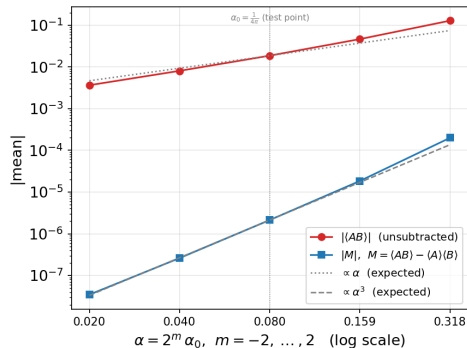


$F_2(Q^2)$: resolved with limited statistics



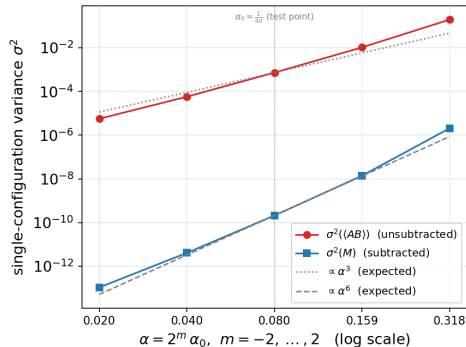
- Isotropic box: $N = 200$, four momentum shells, every point resolved.
- Anisotropic box: ten points, reaching the new lowest point $F_2(Q^2=0.121) = +4.12(70) \times 10^{-6}$.
- Grey: representative extrapolation models (illustrative); a controlled $Q^2 \rightarrow 0$ extrapolation needs lower Q^2 .

The coupling scan: estimator scaling verified in pure QED



means

- The subtracted mean M follows α^3 , as predicted; the unsubtracted $\langle AB \rangle$ is dominated by its $\mathcal{O}(\alpha)$ lower-order piece.
- The subtracted variance follows α^6 ; the subtraction cancels noise by factors 600 to 8500, so $\text{StN} \approx \sqrt{N}$. (Representative momentum; grey lines: expected scaling.)



single-configuration variance

Next steps: what carries over, what changes for HLbL

QCD + quenched QED

- 1 Expected variance scaling $\sigma^2 \sim \alpha^4/N$ (quark loops correlate through gluons); to be validated numerically by the same coupling-scan method.

- 2 Three QCD ensembles available ($M_\pi = 420$ MeV):

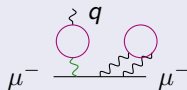
a [fm]	lattice	$M_\pi L$
0.049	64×32^3	3.3
0.075	48×24^3	3.8
0.098	32×16^3	3.3

- 3 Goal: the continuum-extrapolated fully connected HLbL.

Dynamical QCD+QED

- 1 Same expected variance scaling.
- 2 C* QCD+QED ensembles available (RC*):
 $M_\pi = 360\text{--}500$ MeV, $\alpha_R \approx \alpha_{\text{em}}$ and $\approx 5\alpha_{\text{em}}$
[rcstar.gitlab.io/ensembles].

- 3 Automatic inclusion of the leading disconnected HLbL contribution:



- 4 **The goal: fully connected + leading disconnected HLbL in fully dynamical QCD+QED with C* boundary conditions.**

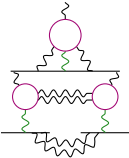
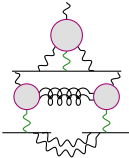
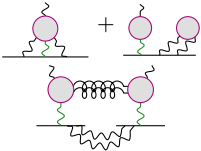
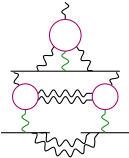
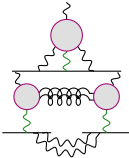
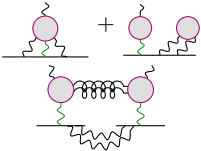
Summary

- 1 The two-current subtraction method is tested in pure QED against the exact Laporta-Remiddi benchmark: $F_2(Q^2)$ resolved (lowest point $F_2(0.121) = +4.12(70) \times 10^{-6}$ vs. $+6.03 \times 10^{-6}$).
- 2 The estimator and its variance are analyzed and validated numerically in QED: signal $\propto \alpha^3$, subtracted variance $\sim \alpha^6$; noise control is estimator design.
- 3 Next: the continuum-extrapolated fully connected HLbL in QCD + quenched QED (three ensembles ready), then connected + leading disconnected in dynamical QCD+QED with C^* : a systematically independent lattice determination of a_μ^{HLbL} .

Computations on CSCS Eiger (Alps).

Backup: estimator, signal, and variance across the stages

$$\hat{\theta} = \frac{N}{N-1} \left[\frac{1}{N} \sum_{i=1}^N A_i^\# \diamond B_i^\wedge - \left(\frac{1}{N} \sum_{i=1}^N A_i^\# \right) \diamond \left(\frac{1}{N} \sum_{j=1}^N B_j^\wedge \right) \right]$$

	QED	QCD + qQED	dynamical QCD+QED
estimator	$A_i^\# = A[\varphi_i]$	$A_i^\# = A_i^C; (U_i, \varphi_i) = \text{QCD} \times$ quenched $U(1)$	$A_i^\# = A_i^C; (U_i, \varphi_i)$ sampled jointly
signal			
variance			

$B_i^\wedge = \frac{1}{2}[B_i(e) - B_i(-e)]$: line charge-parity projection, free as the $\pm \mathbf{p}$ average; kills the line's odd-photon components configuration by configuration.

$A_i^C = \frac{1}{2}(A[U_i, \varphi_i] + A[U_i^*, \varphi_i])$: loop conjugation average, free via the transposition identity on existing solves; restores Furry pointwise on the loop.

Backup: QED treatments across programmes

	QED treatment	fermions	FVE strategy
Mainz	QED_∞ kernel	clover Wilson	analytic kernel
RBC/UKQCD	QED_L / QED_∞	domain wall	cross-checked setups
this project	QED_C (C^* BCs)	Wilson	no photon zero mode; FVE under study

Backup: wraparound and time-slice placement

- On a periodic torus each propagator leg of forward length ℓ carries a backward image at $T - \ell$: $A_{\bar{\mu}}/A_{\mu}|_{\text{leg}} = e^{-m_{\mu}(T-2\ell)}$.
- Leading contamination at a centred insertion: $R_{\text{wrap}} = 2 e^{-m_{\mu}(T-t_{\text{sep}})}$.
- Naive placement (source/sink on the temporal boundaries): wrap distance 1, $R_{\text{wrap}} \approx 150\%$, signal buried. This was diagnosed and the layout corrected.
- Corrected symmetric placement on $T = 24$: $(t_{\text{src}}, t_{\text{op}}, t_{\text{snk}}) = (7, 12, 17)$, $R_{\text{wrap}} = 2e^{-0.2976 \cdot 14} \approx 3\%$.

Backup: sub-sampling benchmark (full table)

M	wall ratio	error ratio (iso)	error ratio (aniso)	cost ratio (iso)
4	0.048	1107	645	5.9×10^4
8	0.057	862	447	4.2×10^4
16	0.073	642	159	3.0×10^4
32	0.101	246	172	6.1×10^3
64	0.168	173	116	5.0×10^3
128	0.298	104	43	3.2×10^3

- Validated against production at $M = V_3$.

Backup: $Q^2 \rightarrow 0$ extrapolation

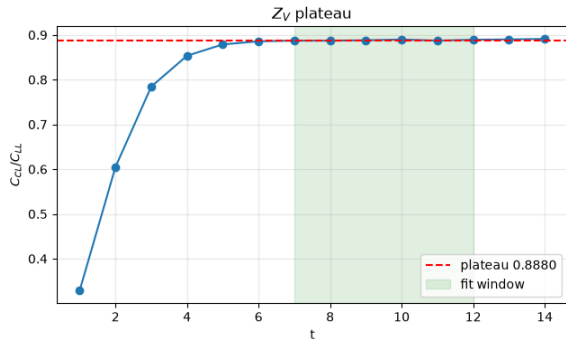
fit	data	$F_2(0) \times 10^6$	class
linear / quadratic	isotropic	+2.0 to +2.3	polynomial (primary)
quadratic	anisotropic (10 pts)	+4.0 (4)	polynomial
monopole / dipole	isotropic	+3.1 / +7.7	pole shapes (bracket only)

- QED $F_2(Q^2)$ has a branch cut at $Q^2 = -(2m_\mu)^2$, not a pole: polynomials are the justified fit class.
- Statistical errors on the data are at the few-percent level of the target; the spread above is model dependence plus volume dependence.
- Bottom line: the computation is extrapolation-limited, not statistics-limited.

Backup: extrapolation models (full table)

fit	data	$F_2(0) \times 10^6$	χ^2/dof	class
linear (all 4)	iso	+2.01 (12)	0.49	polynomial (primary)
linear (2 lowest)	iso	+2.15	n/a	polynomial (primary)
quadratic	iso	+2.34 (37)	0.11	polynomial (primary)
monopole $M^2=(2m)^2$	iso	+3.11 (16)	3.66	pole (bracket only)
dipole $M^2=(2m)^2$	iso	+7.71 (39)	3.44	pole (bracket only)
linear	aniso (10 pts)	+2.60 (15)	n/a	polynomial
quadratic	aniso (10 pts)	+4.01 (44)	n/a	polynomial
combined linear	iso + lowest aniso	+2.07 (12)	n/a	mixes volumes
combined quadratic	iso + lowest aniso	+2.68 (34)	n/a	mixes volumes

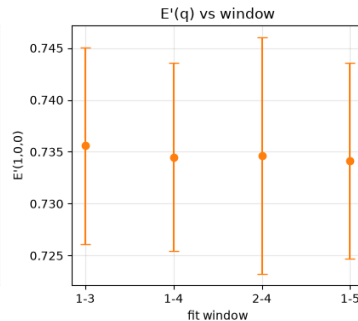
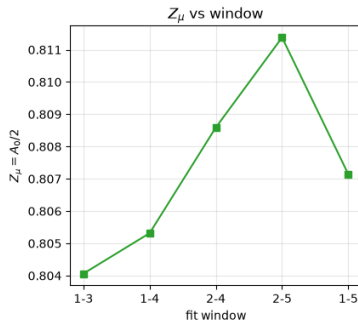
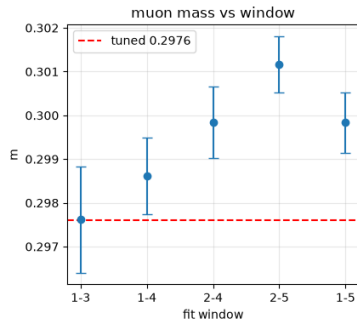
Backup: normalization and renormalization



m (muon)	0.2976 (12)	(tuned 0.2976)
Z_μ	0.8041	($A_0/2$)
Z_V	0.8880 (7)	(plateau $t = 7-12$)
$Z_\mu Z_V^3$	0.5631	
aniso: m, Z_μ	0.2944, 0.7961	

- Fit-window scans flat for $m, Z_\mu, E'(q)$.
- Leptonic multiplicity $1/3$ inside F_2^{raw} .

Backup: fit-window stability



Backup: continuum cost projection

ens	lattice	iso/cfg [node-h]	iso $N=100$ [node-h]	aniso/cfg [node-h]	aniso $N=100$ [node-h]
A	24×8^3	0.57 (meas.)	57	1.58 (meas.)	158
B	32×10^3	2.9-3.6	290-360	8-10	800-1000
C	32×12^3	8.7-13	870-1300	24-36	2400-3600
D	40×16^3	61-122	6100-12200	169-337	17000-34000

Backup: implementation and cost: full volume averaging is mandatory

- Wall time is dominated by the $2V_3+1$ propagator solves per configuration: cost $\sim V_3 \cdot V_4$. Measured anchor: 0.57 node-h per configuration on the coarse box.
- The tempting $10\times$ saving: estimate the source loop stochastically on a random subset of M points. Measured verdict: sub-sampling breaks the M_1 - M_2 correlation, and the error on the subtracted observable inflates by 10^2 to 10^3 (both geometries, all M ; backup table). Cost-to-precision worsens by three orders of magnitude.
- Conclusion, carried to HLbL directly: the loop must be averaged over the full volume; production is budgeted accordingly.

Backup: continuum limit: settled, costed, measurement pending

- Normalization and renormalization under control on the same ensembles: m_μ , Z_μ , $Z_V = 0.8880(7)$ measured, overall normalization derived, no external inputs (backup).
- Ensemble chain A-D at fixed $m_\mu L = 2.4$, factor 2 in spacing: generated and tuned (generation costs seconds per configuration in quenched QED).
- Measurement costed from the anchor: $B \approx 300$, $C \approx 10^3$, $D \approx 10^4$ node-h at $N=100$ (isotropic; backup table); single-configuration benchmarks before committing.
- Empirical continuum results: pending; the next production step.

Backup: variance-analysis pipeline and scan method

Variance analysis (the pipeline):

- Expand $\langle \bar{o}^2 \rangle$ over coincidence patterns; three connected structures survive:
 $\langle ABAB \rangle_c + \langle \widehat{AB} \rangle_c^2 + \langle AA \rangle_c \langle BB \rangle_c$.
- Furry's theorem kills every α^4 term: leading variance α^5/N .
- The $\pm \mathbf{p}$ average ($= e \rightarrow -e$ on the muon line, Blum et al. 2015) projects the line onto even photon number: predicted to remove the α^5 term, giving $\sigma^2 \sim \alpha^6/N$ and $\text{StN} = \sqrt{N} \alpha^0$.

Numerical verification (the trick):

- The coupling lives only in the photon-field normalization: rescaling the stored configurations by $\sqrt{r}$ runs the identical pipeline at $r\alpha$ (validated bit-wise).
- κ retuned per coupling: fixed lepton mass, so fitted slopes are pure powers of α .
- $r = \frac{1}{4} \dots 4$, $N = 50$ each, same Gaussian draws across couplings; ≈ 230 node-h total.

Backup: project status and to-do

item	status / plan
✓ subtraction method, form factor projection	exact and measured checks
✓ wraparound control (symmetric placement, $T = 3L$)	contamination $\approx 3\%$
✓ estimator study	variance scaling in α verified by scan
✓ leptonic $F_2(Q^2)$, coarsest ensemble, two geometries	$N = 200$ / $N = 100$
□ leptonic continuum: ensembles B, C, D	costed; benchmarks first
□ lower Q^2 / volume control	C^* momenta, larger L
□ stage 2: QCD + quenched QED quark loop	3 spacings, $M_\pi = 420$ MeV
□ stage 3: dynamical QCD+QED on RC^* C^* ensembles	disconnected automatic

Backup: coupling-scan exponents, full table

quantity	all r	$r \leq 1$	$r \geq 1$	physical power
signal F_2	2.16	1.8	2.4	α^3 at LO
σ^2 subtracted	3.92	3.3	4.7	$\alpha^5 \rightarrow \alpha^6$ crossover
σ^2 unsubtracted	1.59	1.25	2.0	$\alpha^3 \rightarrow \alpha^4$

- Bookkeeping: physical power = scan slope +1 (signal) or +2 (variances); the manually inserted photon line does not rescale.
- Local per-octave slopes of σ^2 (subtracted): 3.2, 3.7, 4.2, 5.2: the physical coupling sits inside the α^5 - α^6 crossover.
- Line of constant physics: κ retuned per coupling, $am = 0.297$ - 0.303 across a factor 16 in α .

Backup: line of constant physics (tuned κ)

r	tuned κ	am	fixed $\kappa = 0.12412$:
1/4	0.117311	0.2973(8)	am drifts 0.13 to 0.91 (QED self-energy at large coupling)
1/2	0.119356	0.3028(15)	
1	0.124120	0.2992(31)	
2	0.133848	0.3009(64)	
4	0.155689	0.3033(145)	

- At fixed κ the mass drift injects kinematic exponentials into every raw variance comparison; the tuned chain (secant on the two-point function, seconds per step) removes them, so fitted slopes are pure powers of α . The fixed- κ scan is kept as a systematics check.