

Mainz HVP update

Arnau Beltran Martínez for the Mainz lattice group

Johannes Gutenberg-Universität Mainz



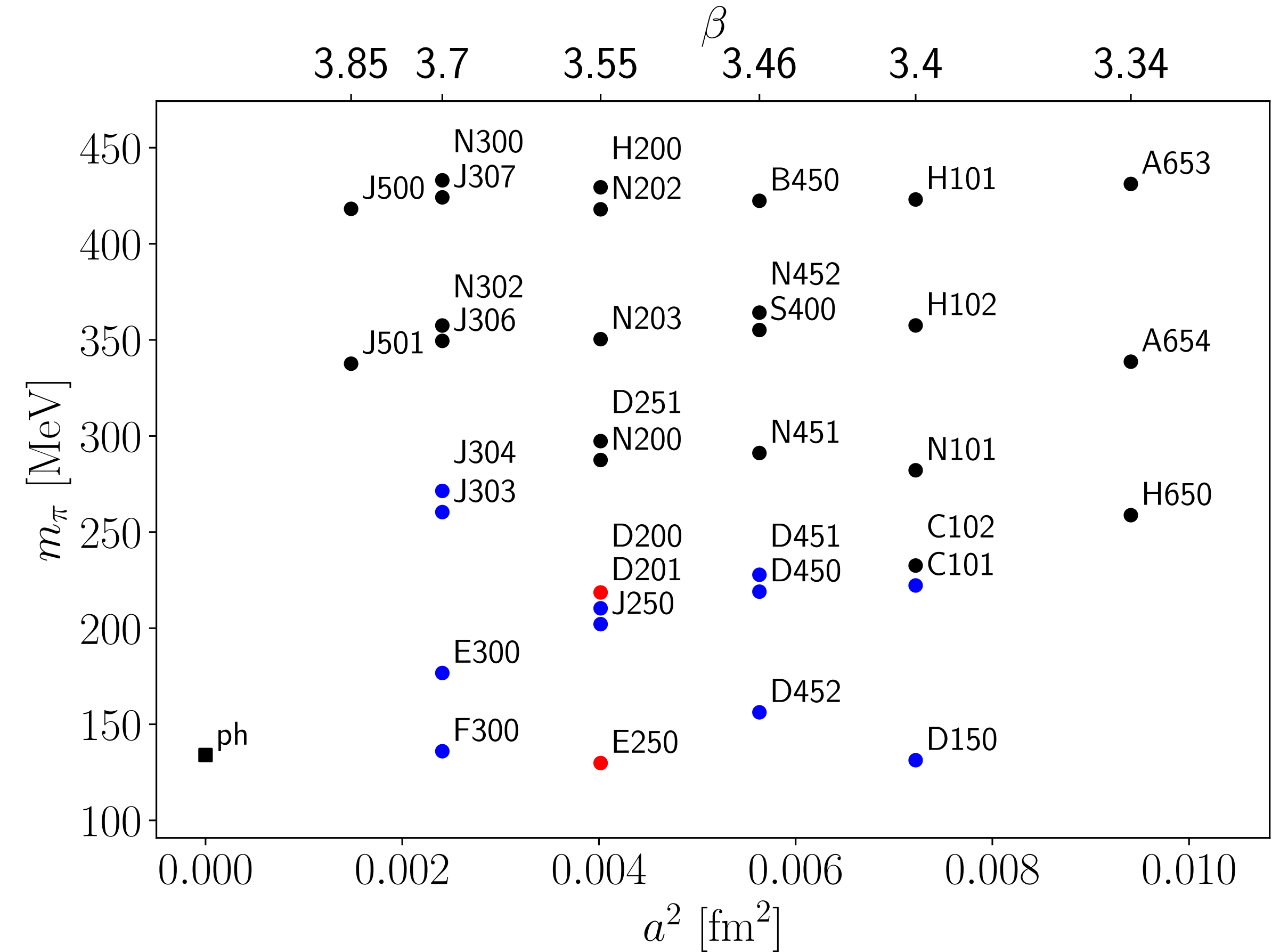
August 2026

$g - 2$ Theory Initiative - University of Connecticut

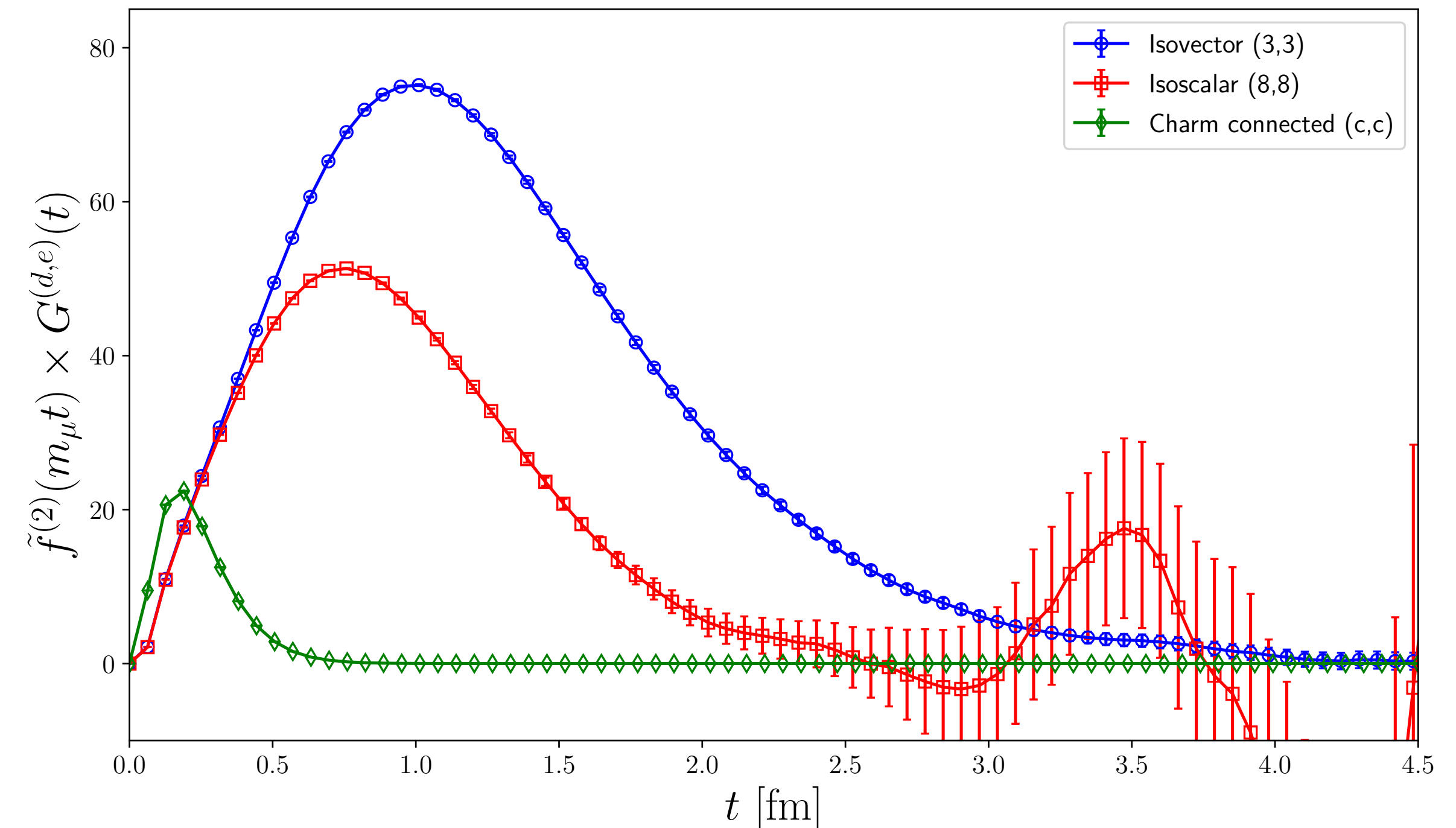
- Introduction
- IsoQCD projects
 - $\Delta\alpha^{\text{had}}(Q^2)$
 - Higher-order HVP in a_μ
 - Other lepton flavors
- Isospin breaking corrections

$$G^{(\gamma,\gamma)}(t) = -\frac{1}{3} \int d^3\mathbf{x} \langle j_k^\gamma(t, \mathbf{x}) j_k^\gamma(0) \rangle$$

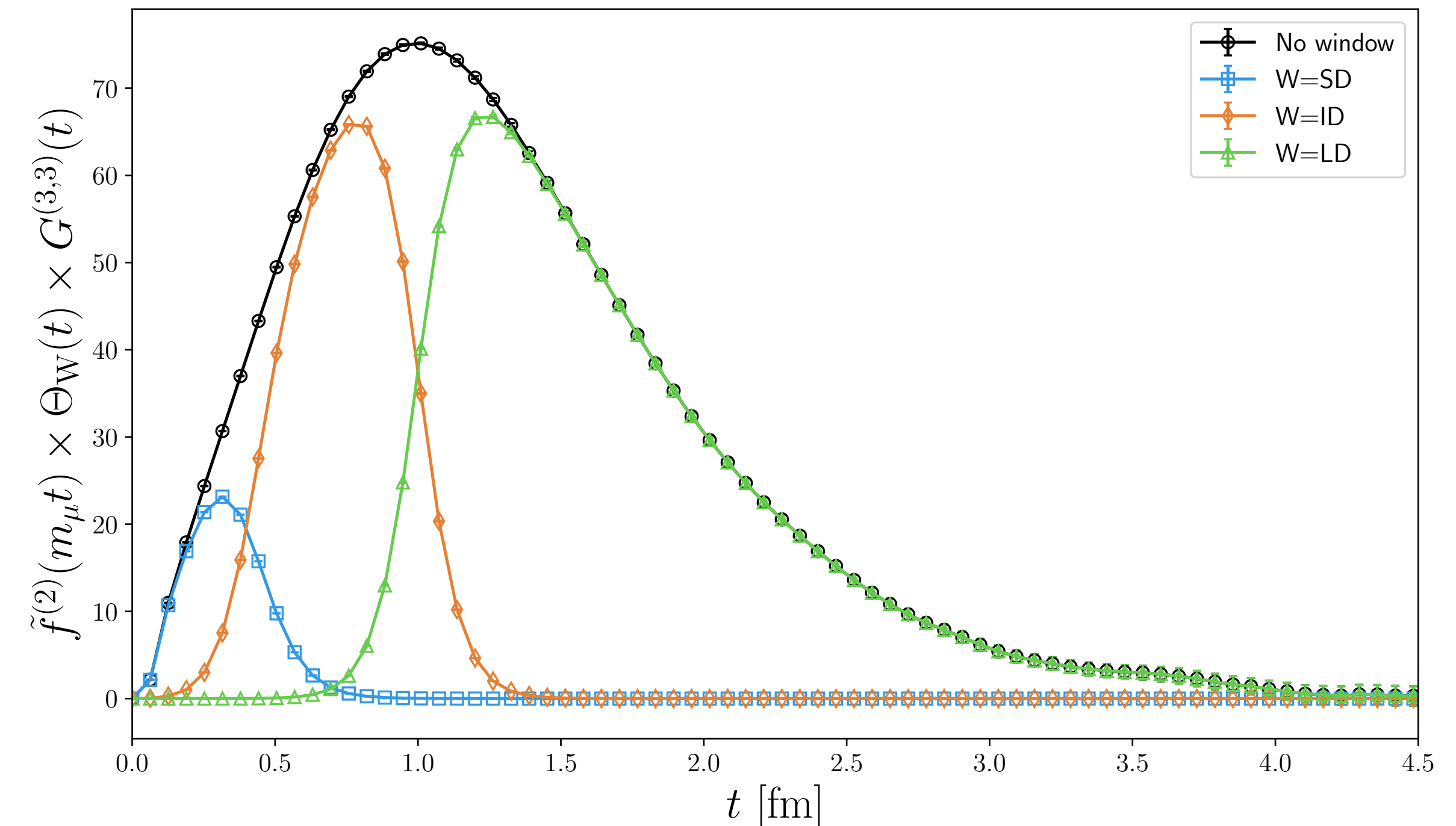
- Large **data set** already created (CLS)
 - 2 chiral trajectories: $\text{tr}M = \text{ct}$ and $m_s = m_s^{\text{ph}}$
 - 3 ensembles at physical pion mass
- Current efforts in improving and extending the data set
- Specific efforts to mitigate the StN problem in the long distance tail
 - We have performed **LMA** on 14 ensembles
 - **Spectroscopy analysis** on E250 and D200



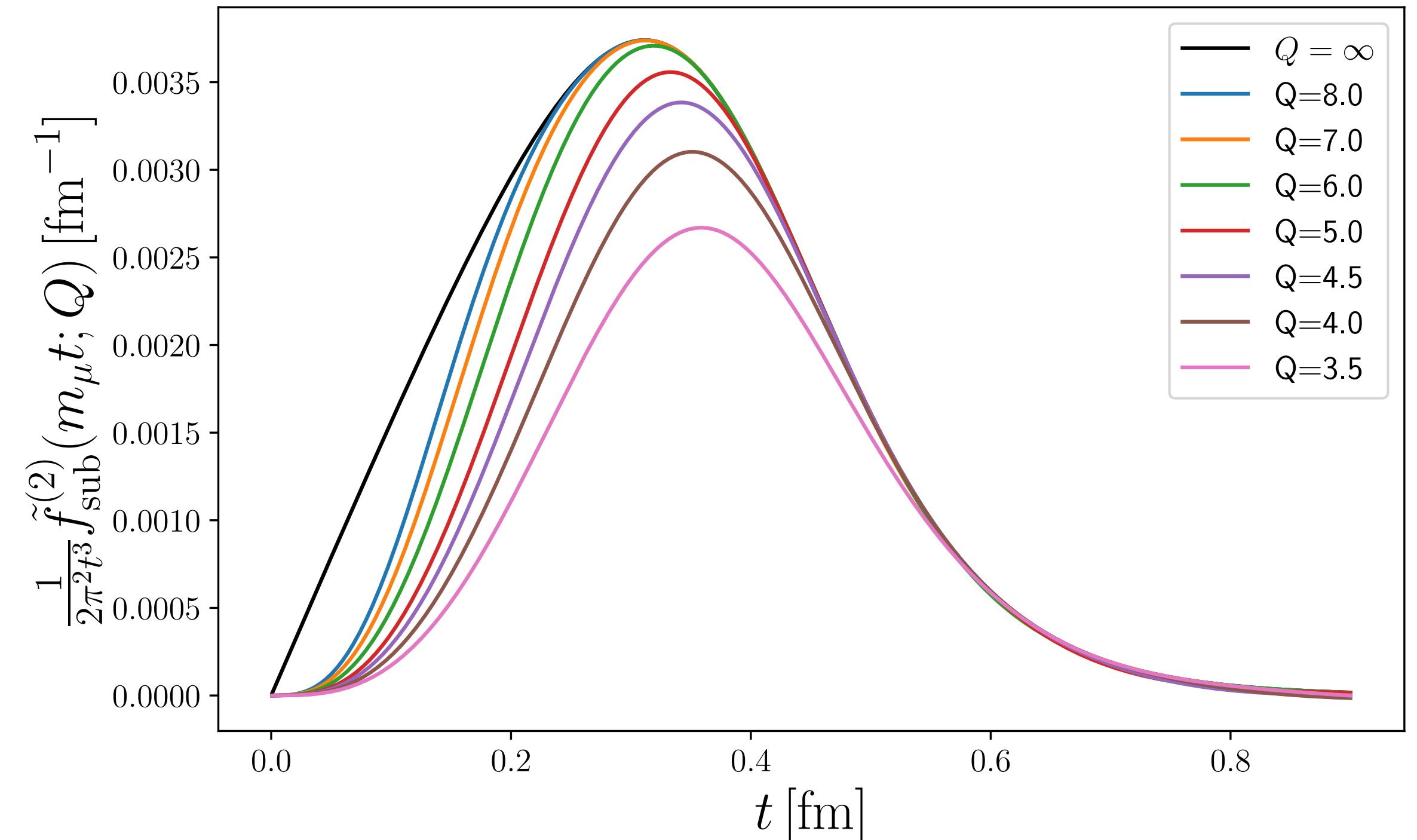
- Isospin basis decomposition (3,3 8,8 c,c ...)
- Windowing procedure (SD, ID, LD)
- Cancellation of log-enhanced cutoff effects by subtracting the most short-distance $\mathcal{O}(t^4)$ physics of the integrand
- Bounding method and reconstruction
- 2-step computation of finite volume effects
 - Correction to V_{ref} before extrapolation to ph point (HP & MLL methods)
 - Correction from V_{ref} to ∞ in the continuum



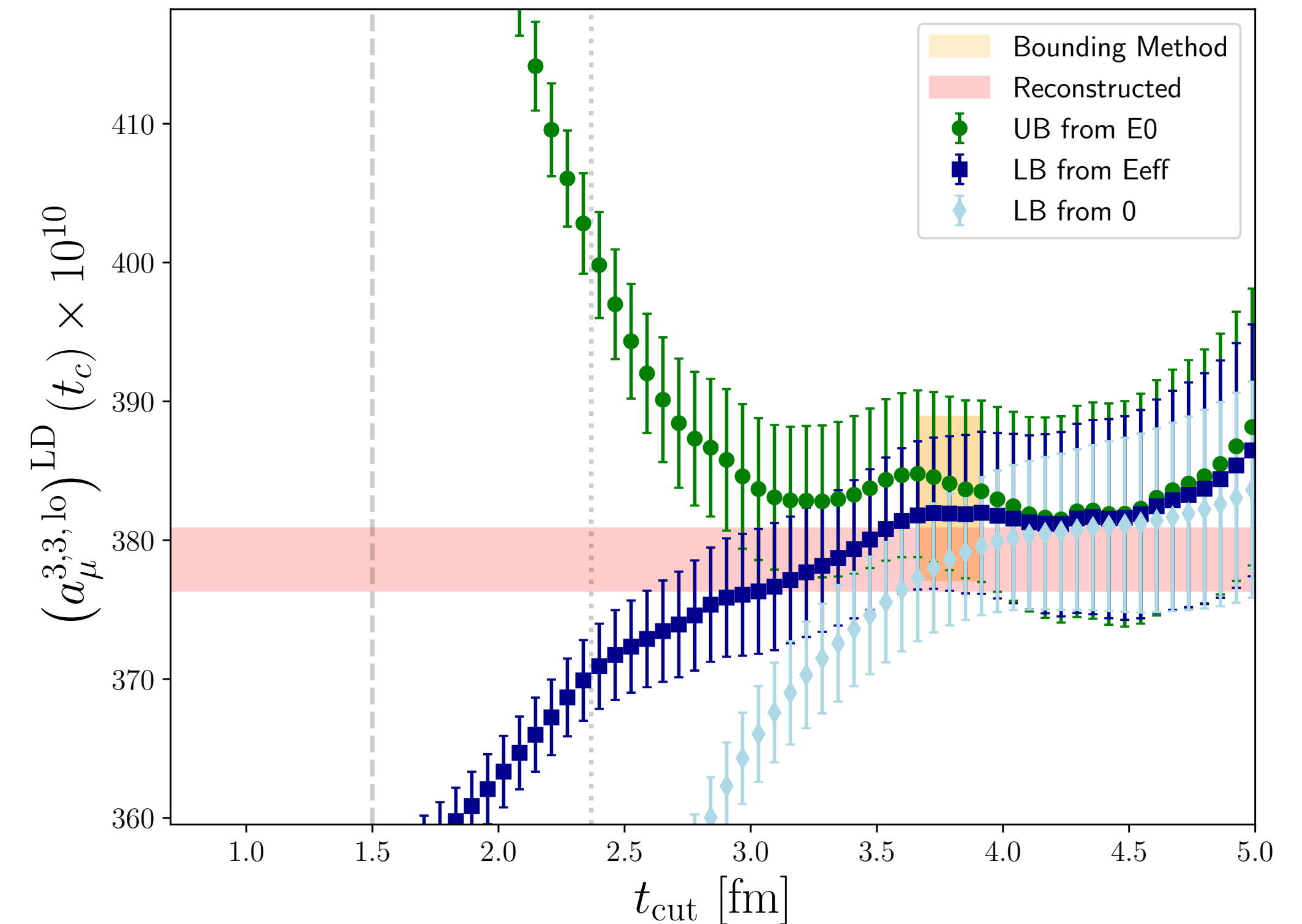
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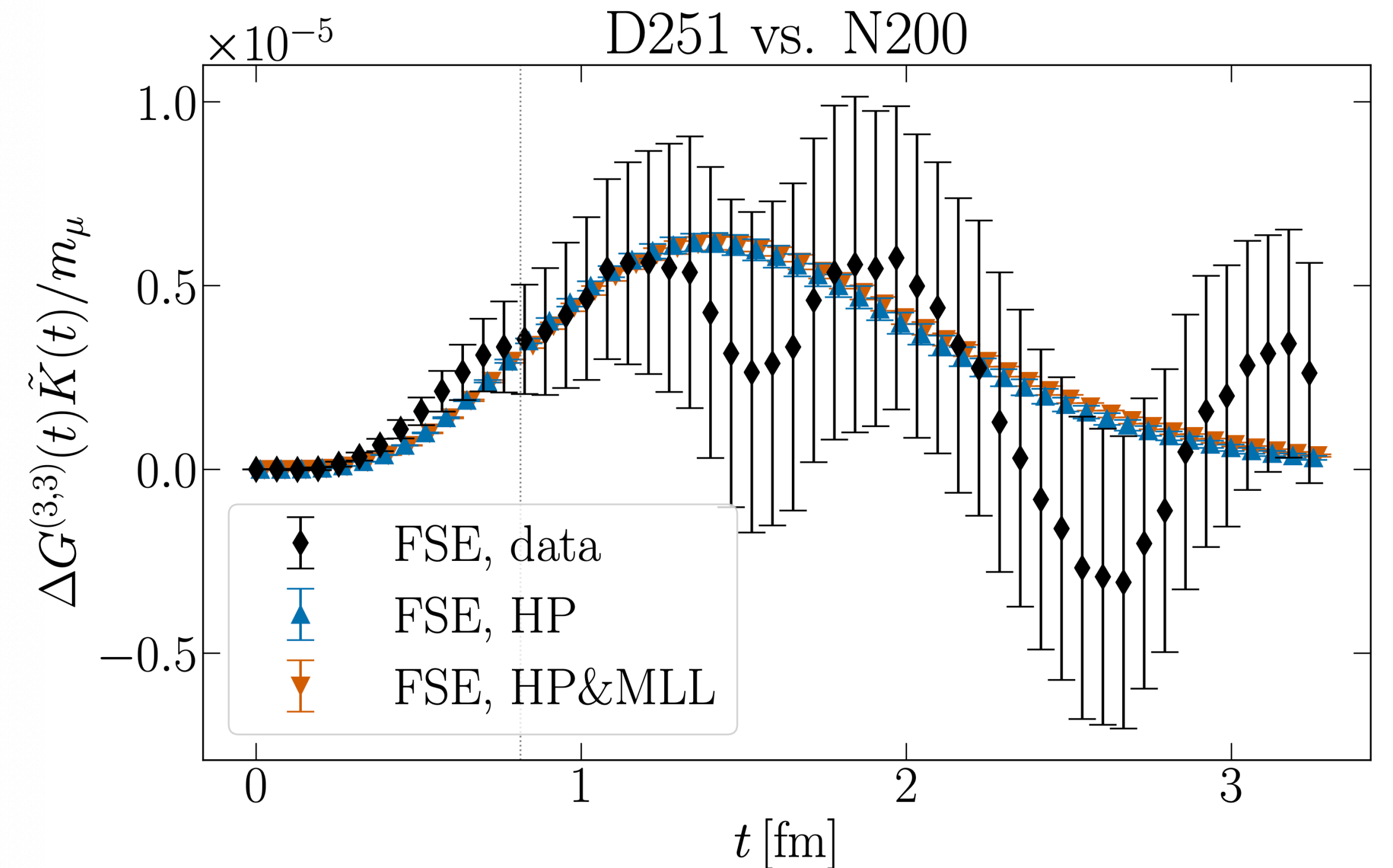
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$$\mathcal{O}^{\text{hvp}}(Q) \sim \int dt w(t; Q) G(t)$$

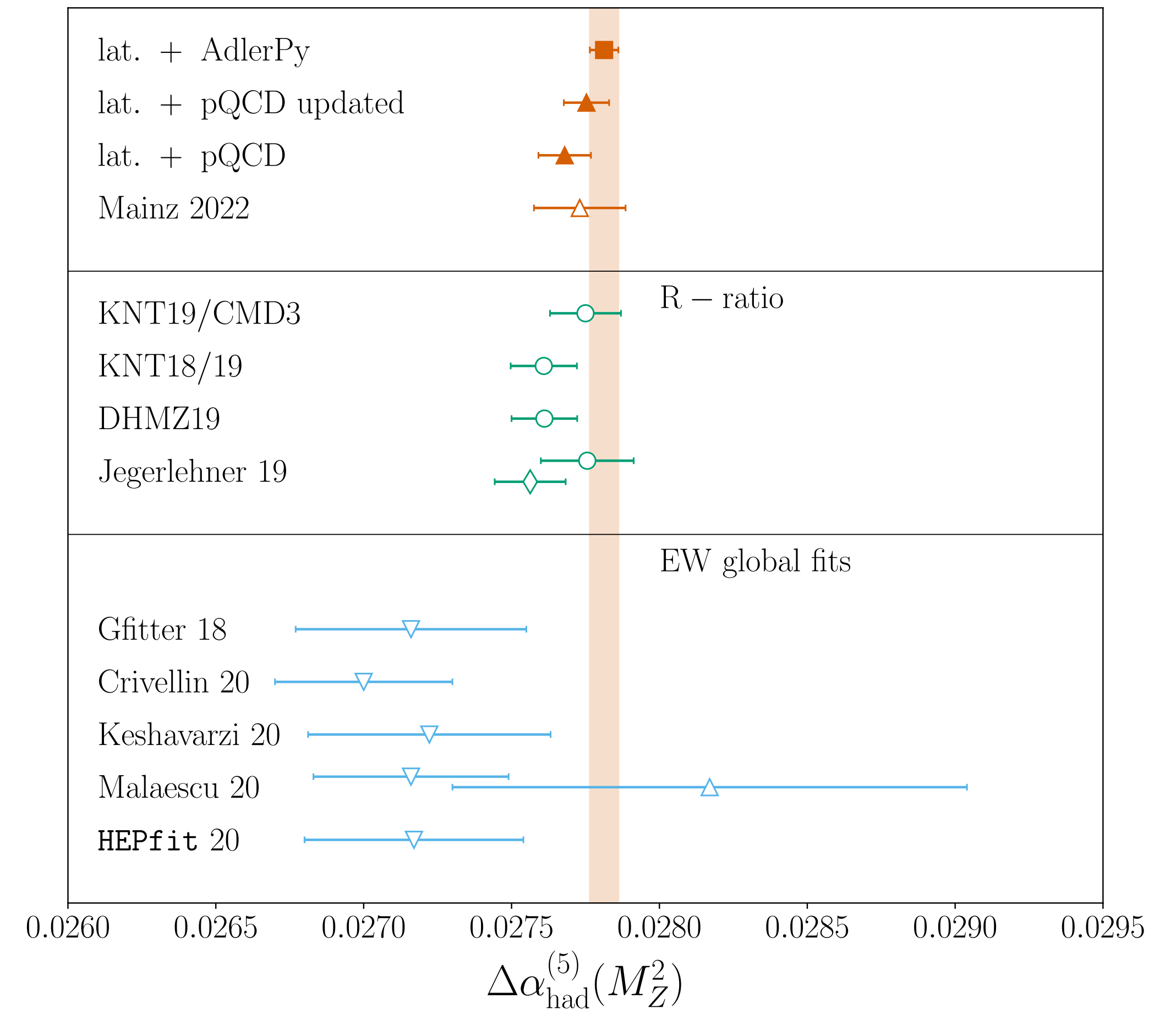
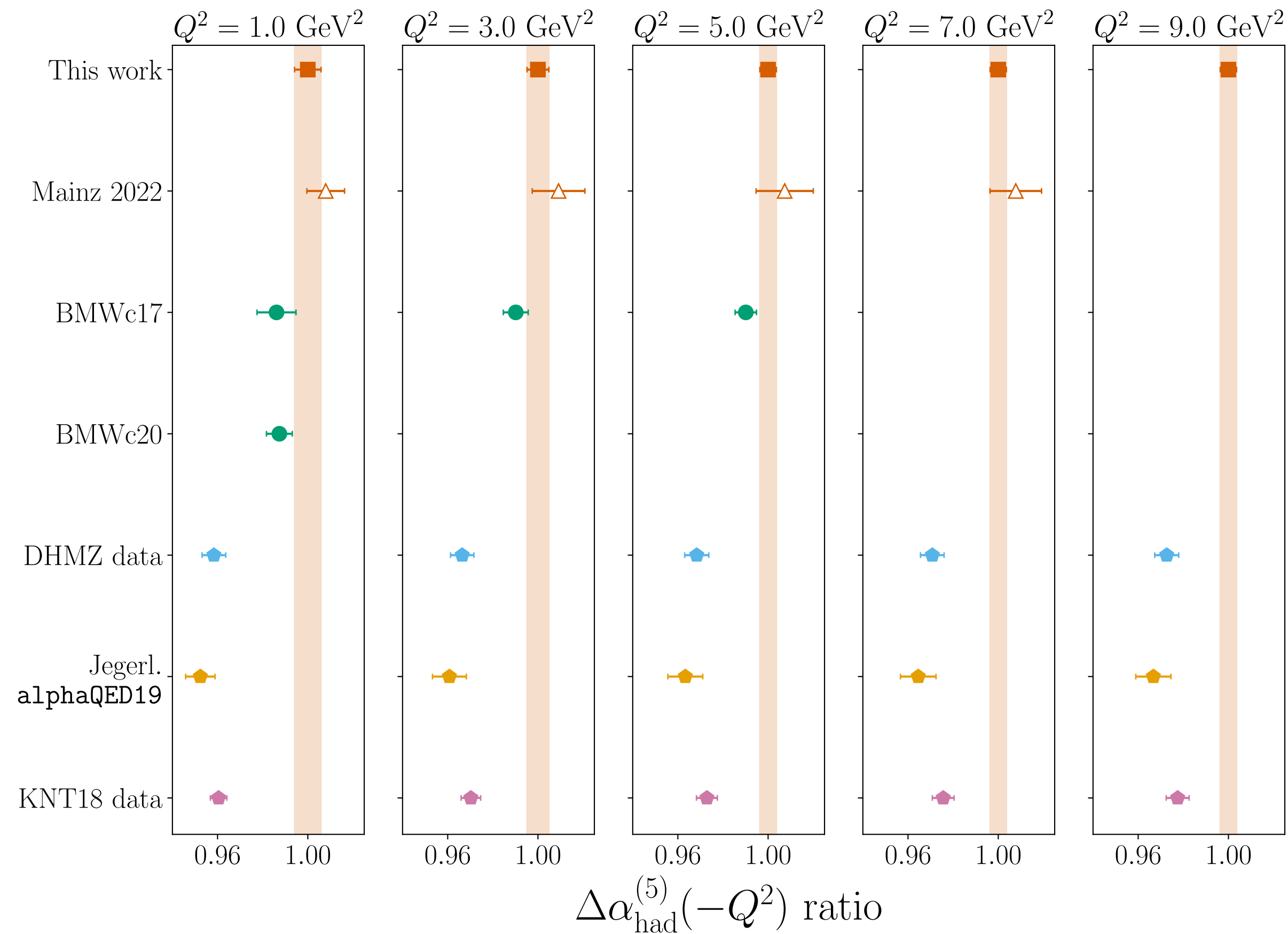
- We probe **observables sensitive to the HVP**, each weighting a different region of the hadronic spectrum / Euclidean momentum
 - ▶ **Long distance** - $a_\mu^{\text{hvp, lo}}$, $a_e^{\text{hvp, lo}}$, $\Pi'(0)$
 - ▶ **Short distance** - $\Delta\alpha_{\text{had}}(M_Z^2)$, $\Delta(\sin^2 \theta_W)_{\text{had}}$, $a_\mu^{\text{hvp, nlo}}$, $a_\tau^{\text{hvp, lo}}$ (*can be computed to high precision*)
- Compare **data- and lattice-based determinations** across these observables to test their (in)consistency

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The running of the QED coupling

$$\Delta\alpha^{\text{had}}(Q^2)$$

See A. Conigli's talk: Thursday 09:25 - 09:55



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Representations of the *master formula*

TMR

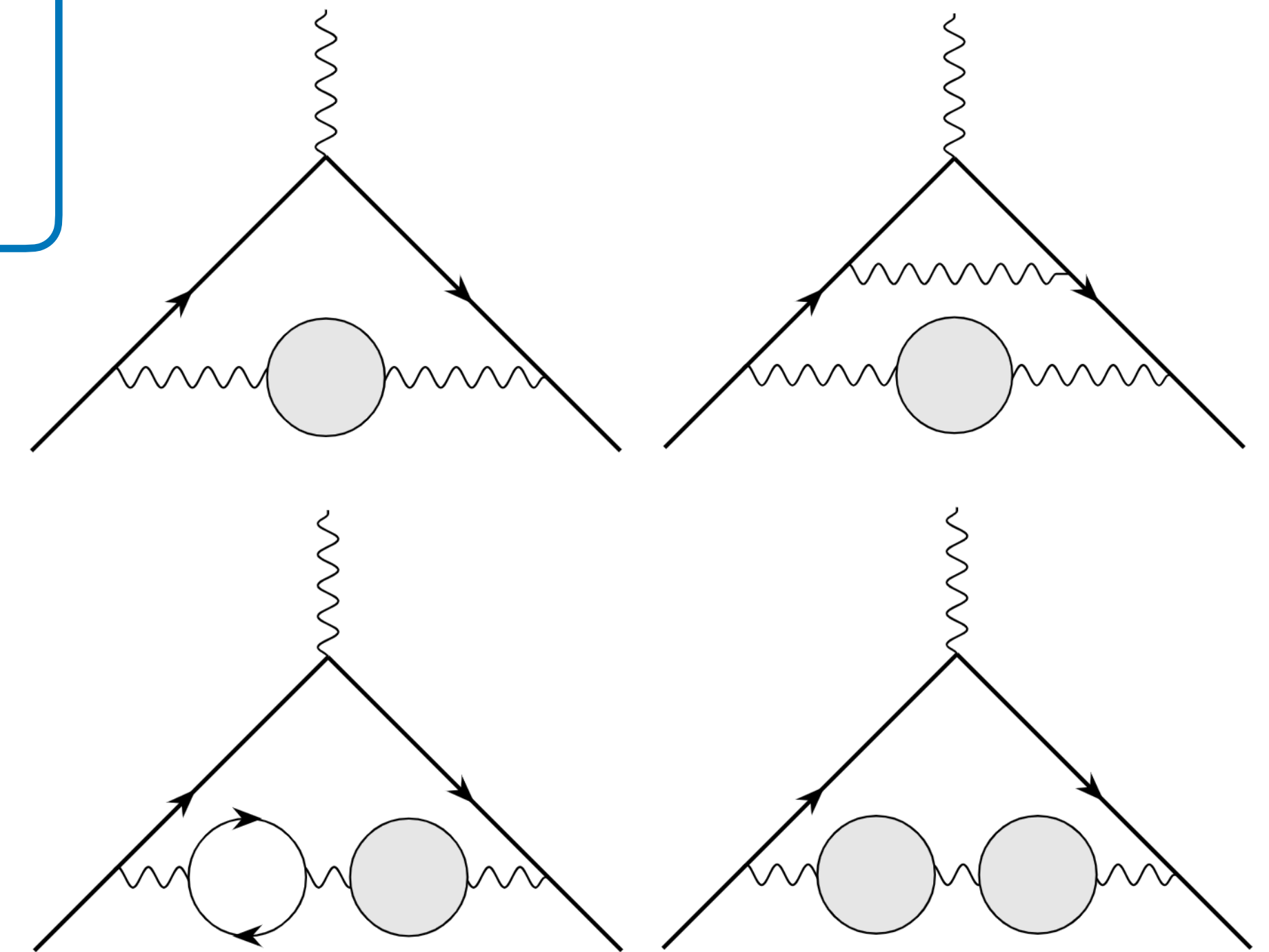
[D. Bernecker and H. B. Meyer 2011]

$$a_{\ell}^{\text{hvp}, (i)} = \left(\frac{\alpha}{\pi} \right)^{2+N_i} \int dt_1 \dots dt_{m_i} \tilde{f}^{(i)}(\hat{t}_1, \dots, \hat{t}_{m_i}) G(t_1) \times \dots \times G(t_{m_i})$$

$$\tilde{f}^{(i)}(\hat{t}_1, \dots, \hat{t}_{m_i}) = \int_0^\infty d\hat{\omega}^2 \hat{f}^{(i)}(\hat{\omega}^2) \prod_{j=1, \dots, m_i} \frac{4\pi^2}{m_{\ell}^2 \hat{\omega}^2} \left[\hat{\omega}^2 \hat{t}_j^2 - 4 \sin^2 \frac{\hat{\omega} \hat{t}_j}{2} \right]$$

$$N_i = \begin{cases} 0 & (i) = (2) \\ 1 & (i) = (4a), (4b), (4c) \\ \dots & \end{cases}$$

$$m_i = \# \text{ } \bigcirc$$



We want analytical control over $\tilde{f}$ for all NLO diagrams

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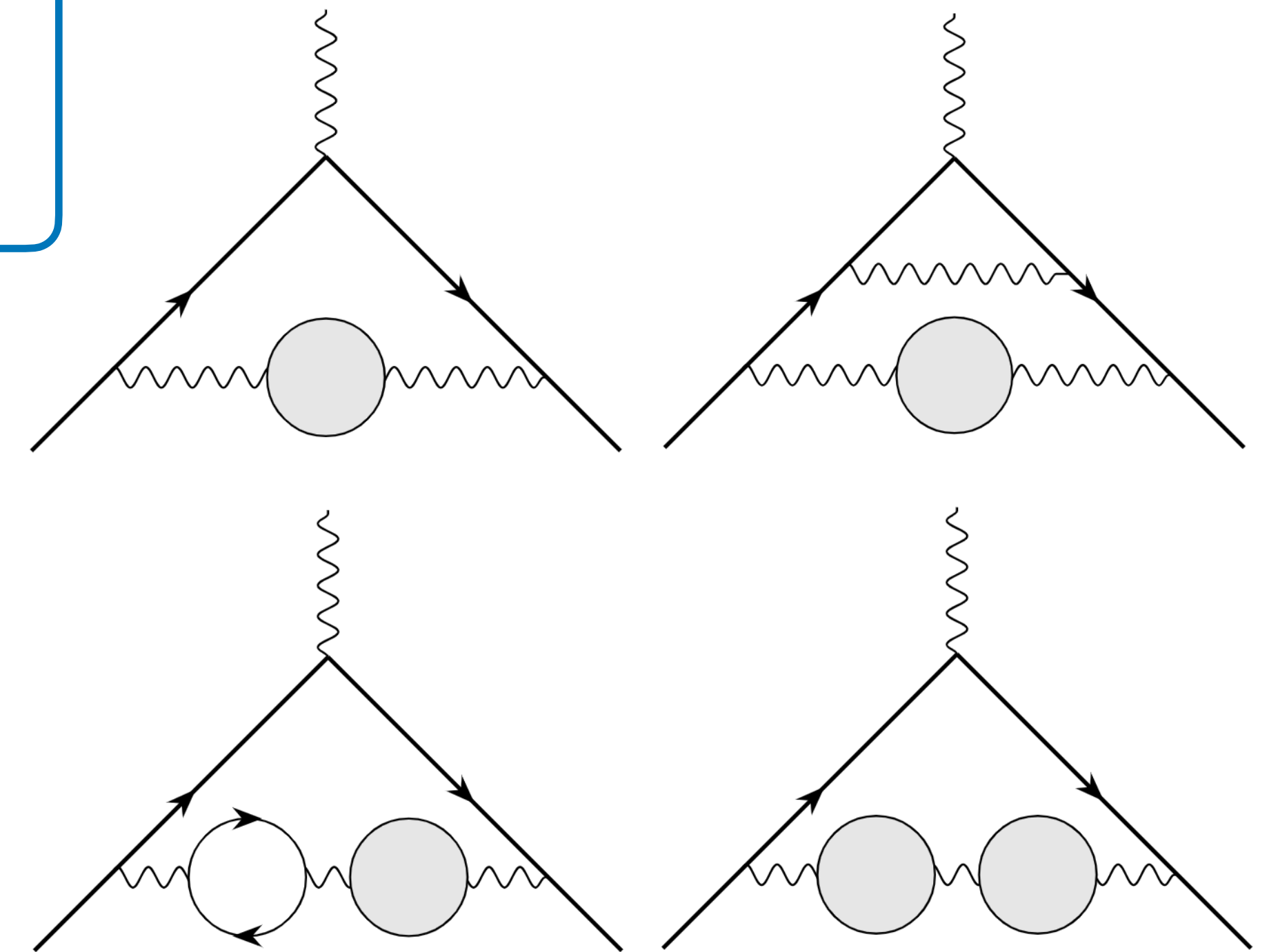
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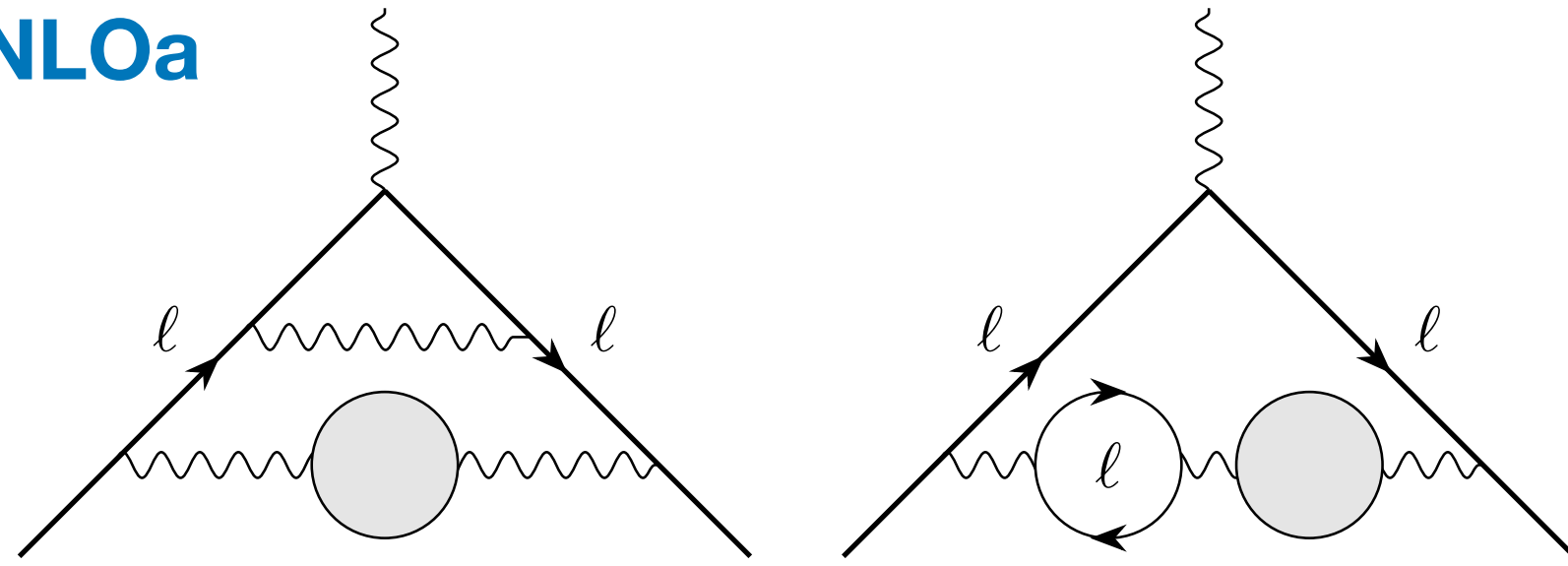


We want analytical control over $\tilde{f}$ for all NLO diagrams

NLOa and NLOb diagrams

TMR

NLOa



[S. Laporta et al. 2024]

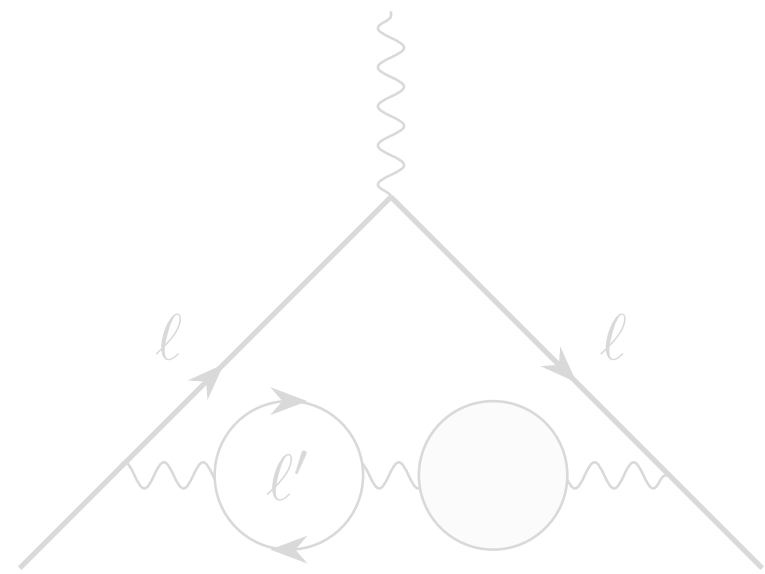
$$\frac{m_\ell^2}{16\pi^2} \tilde{f}^{(4a)}(\hat{t}) \stackrel{\hat{t} \ll 1}{\equiv} \sum_{\substack{n \geq 4 \\ n \text{ even}}} \frac{\hat{t}^n}{n!} \left(a_n + b_n \pi^2 + c_n (\gamma_E + \ln \hat{t}) + d_n (\gamma_E + \ln \hat{t})^2 \right)$$

Dominant piece: $\mathcal{D} \sim \hat{t}^2$

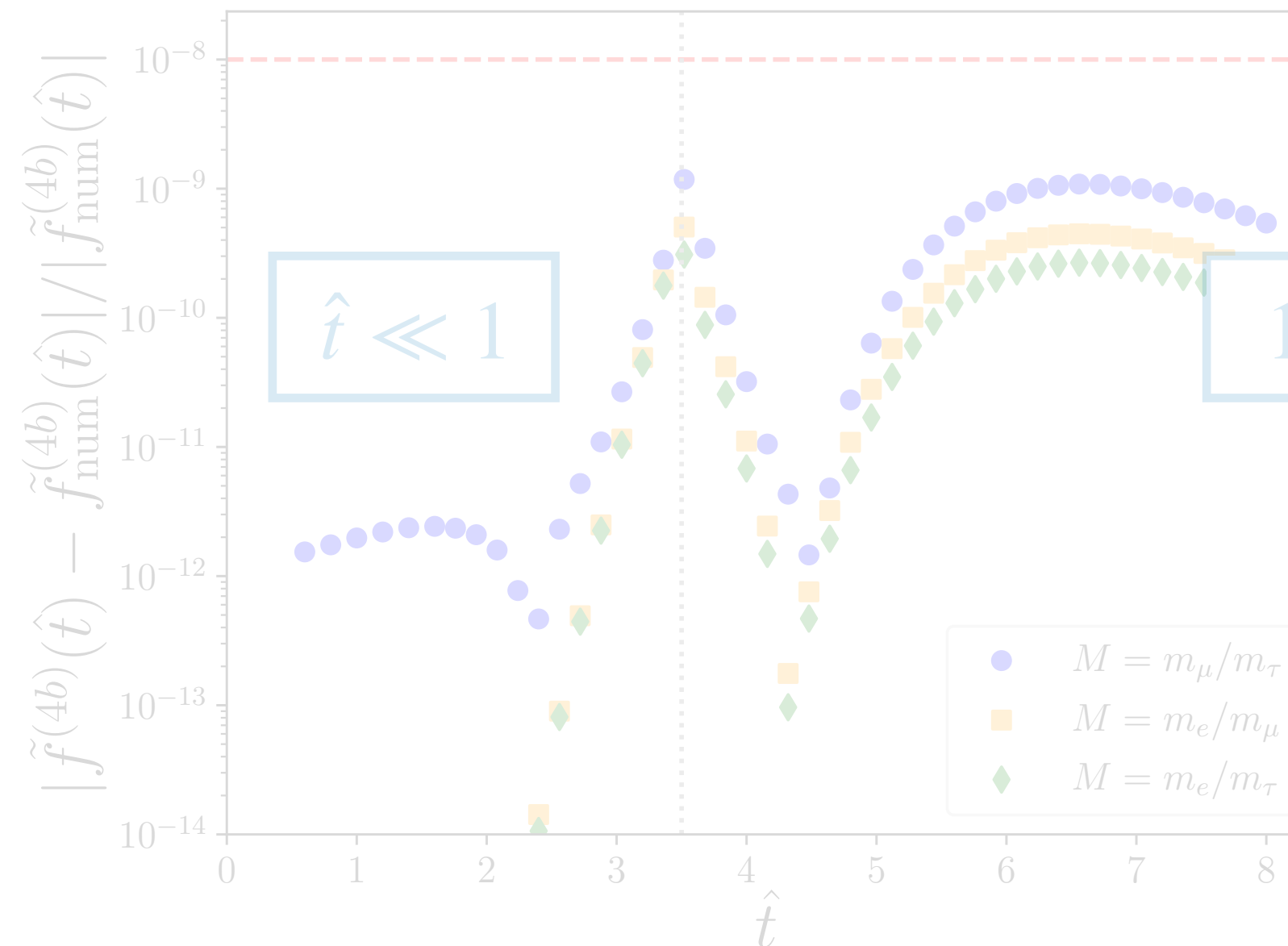
Suppressed piece: $\mathcal{S}$

$$\frac{m_\ell^2}{16\pi^2} \tilde{f}^{(4a)}(\hat{t}) \stackrel{\hat{t} \gg 1}{\equiv} \mathcal{D}(\hat{t}) + \mathcal{S}(\hat{t})$$

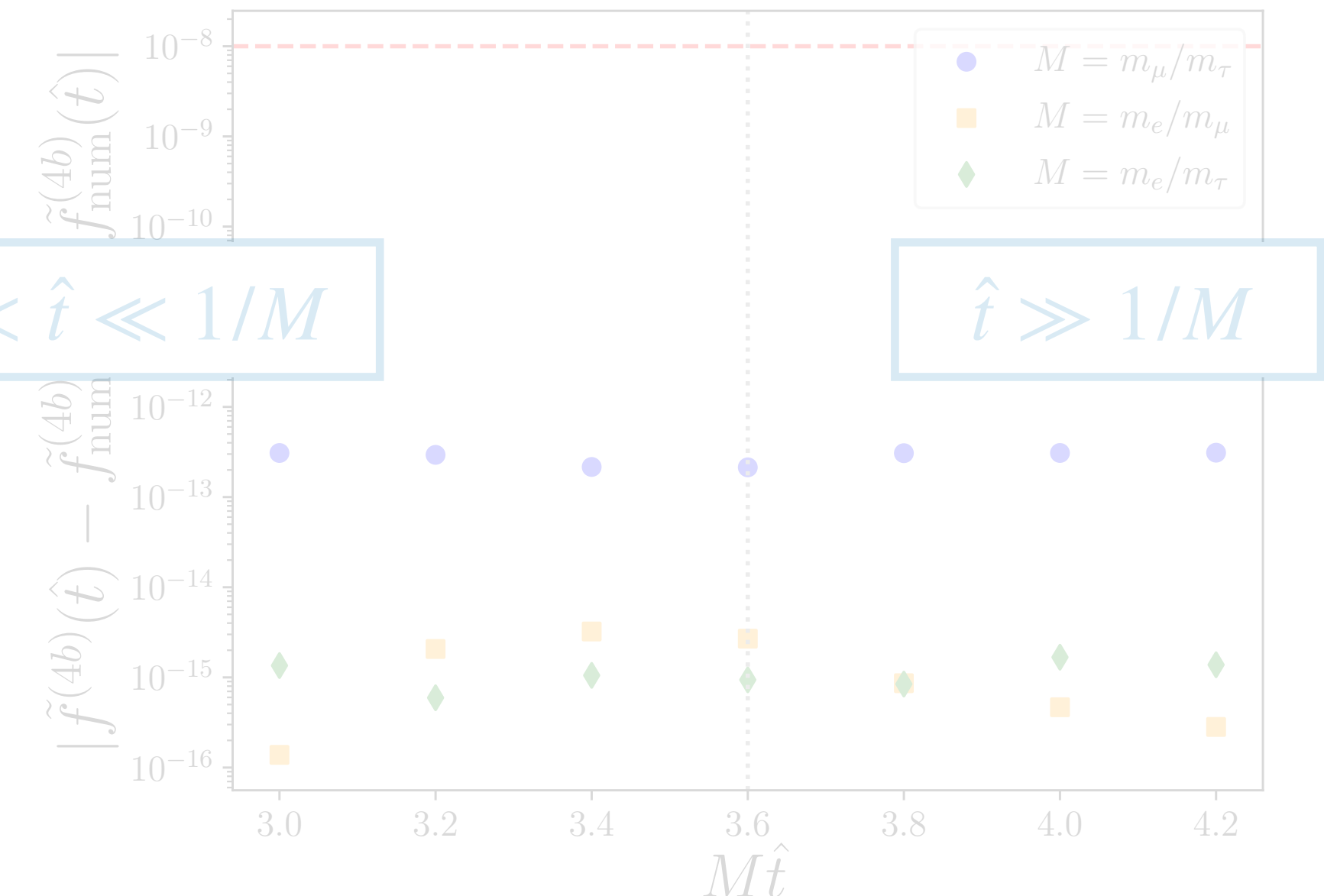
NLOb



$$M \equiv \frac{m_{\ell'}}{m_\ell} \ll 1$$



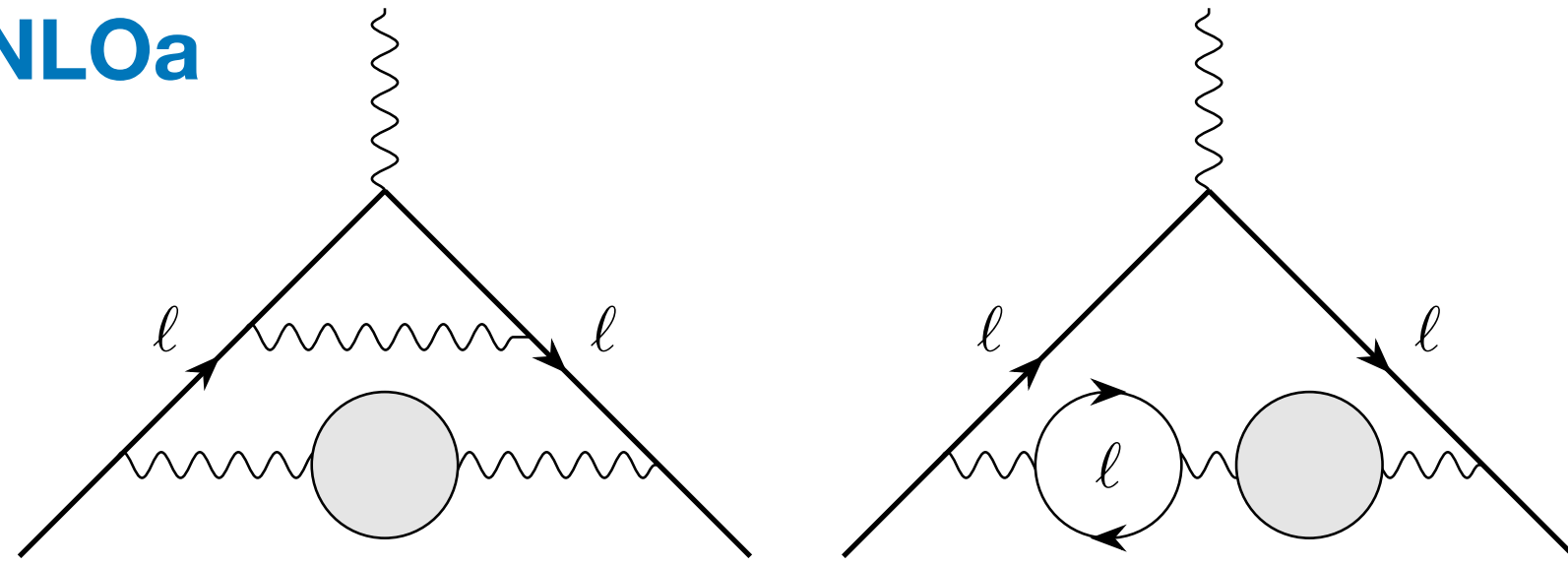
$$1 \ll \hat{t} \ll 1/M$$



NLOa and NLOb diagrams

TMR

NLOa



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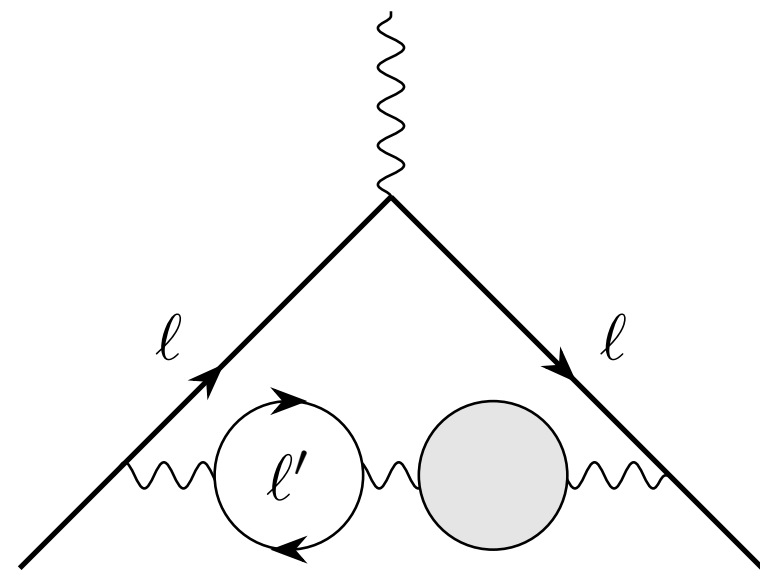
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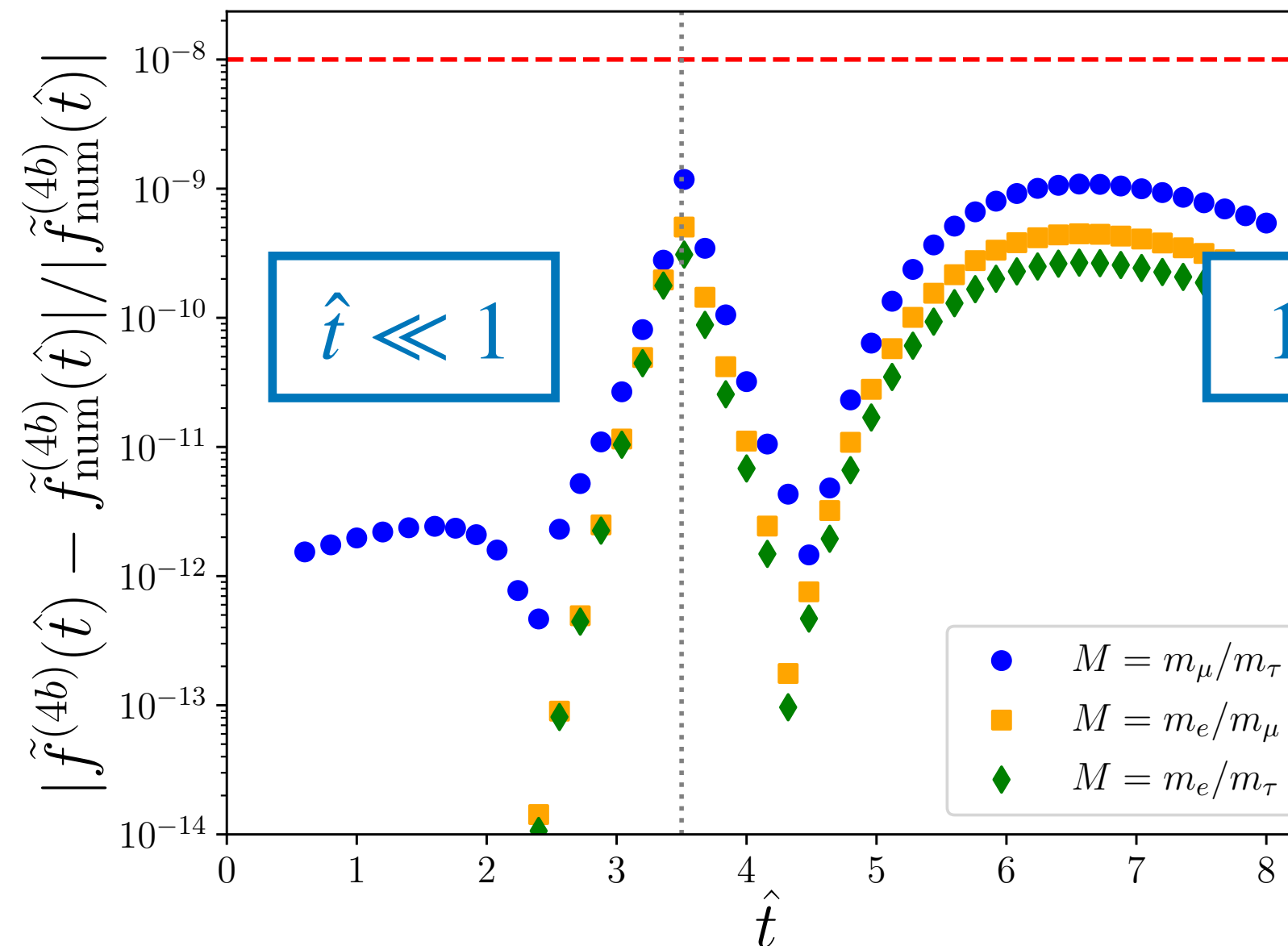
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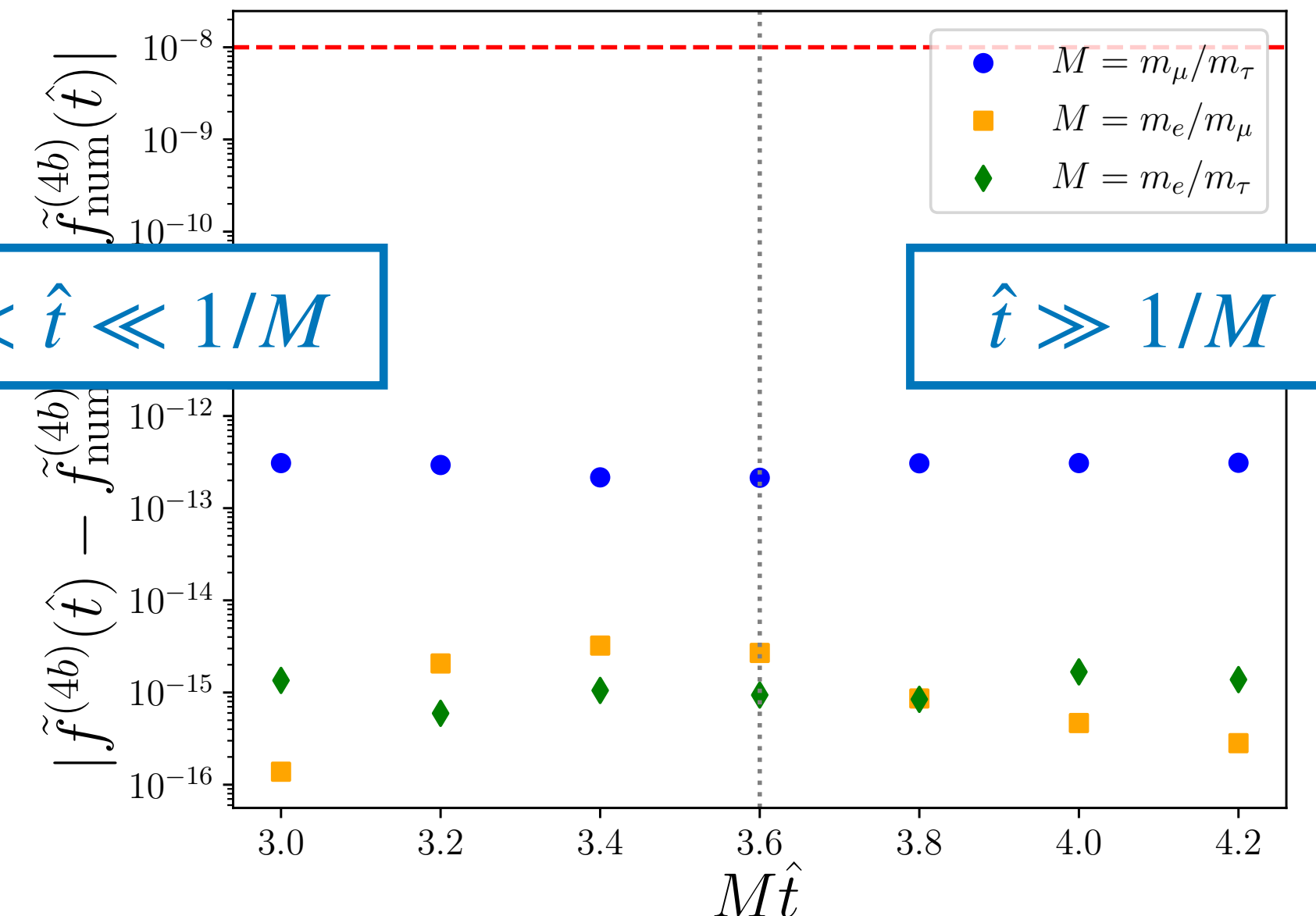
NLOb

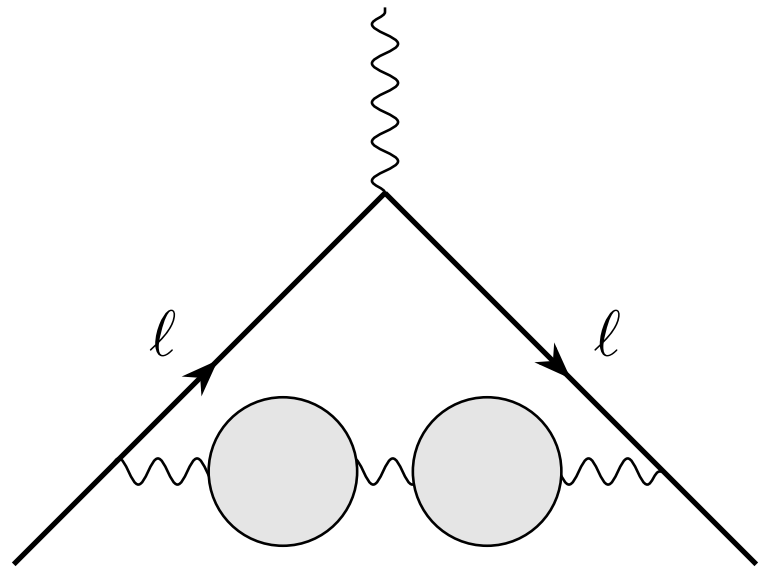


$$M \equiv \frac{m_{\ell'}}{m_\ell} \ll 1$$



$$1 \ll \hat{t} \ll 1/M$$



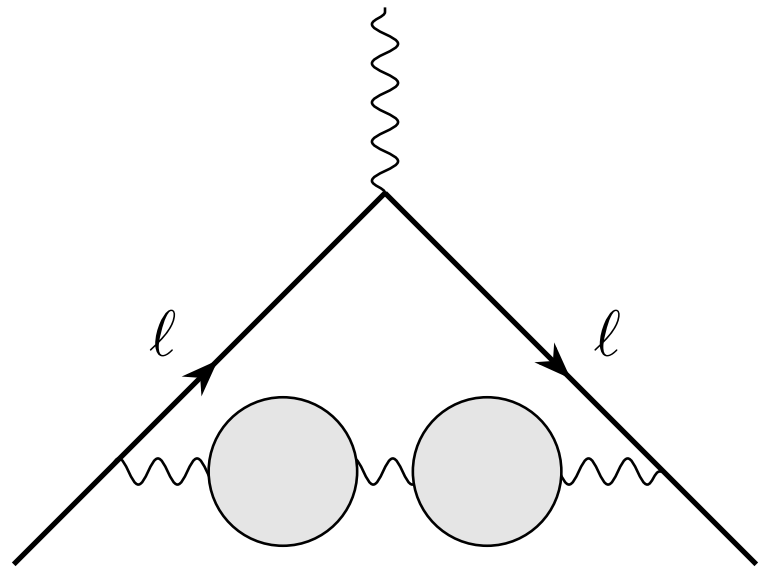


$$\frac{m_\ell^4}{32\pi^4} \tilde{f}^{(4c)}(\hat{t}, \hat{\tau}) = \int_0^\infty \frac{d\hat{\omega}}{\hat{\omega}^3} \hat{f}^{(2)}(\hat{\omega}^2) \left[\hat{\omega}^2 \hat{t}^2 - 4 \sin^2 \frac{\hat{\omega} \hat{t}}{2} \right] \left[\hat{\omega}^2 \hat{\tau}^2 - 4 \sin^2 \frac{\hat{\omega} \hat{\tau}}{2} \right]$$

$$\frac{m_\ell^4}{32\pi^4} \tilde{f}^{(4c)}(\hat{t}, \hat{\tau}) = \frac{\hat{t}^2 \hat{\tau}^2}{4} - 2(1 + \gamma_E) + \hat{t}^2 \mathcal{A}(\hat{\tau}) + \hat{\tau}^2 \mathcal{A}(\hat{t}) + \left[\sum_{\theta=(\hat{t}+\hat{\tau}), |\hat{t}-\hat{\tau}|} - 2 \sum_{\theta=\hat{t}, \hat{\tau}} \right] \mathcal{B}(\theta)$$

$$\mathcal{A}(\theta) = \frac{1}{\theta^2} - \frac{1}{2} + 2(\ln \theta + \gamma_E) - \frac{2}{\theta} K_1(2\theta) + \frac{1}{4} G_{1,3}^{2,1} \left(\theta^2 \middle| \frac{3}{2} \right)_{0,1,\frac{1}{2}}$$

$$\mathcal{B}(\theta) = (1 - \theta^2) \ln \theta + K_0(2\theta) + \frac{1}{2} G_{1,3}^{2,1} \left(\theta^2 \middle| \frac{3}{2} \right)_{0,1,\frac{1}{2}} - \frac{1}{16} G_{2,4}^{2,2} \left(\theta^2 \middle| 1, \frac{5}{2} \right)_{1,2,0,\frac{1}{2}}$$

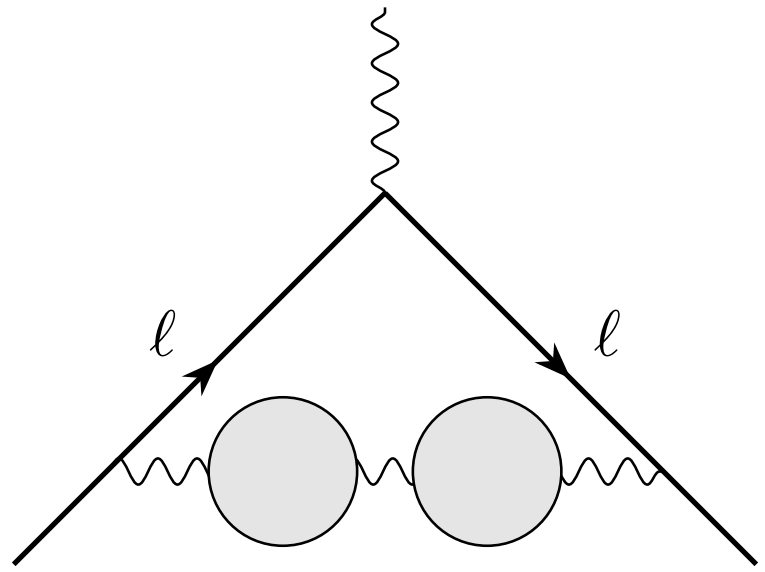


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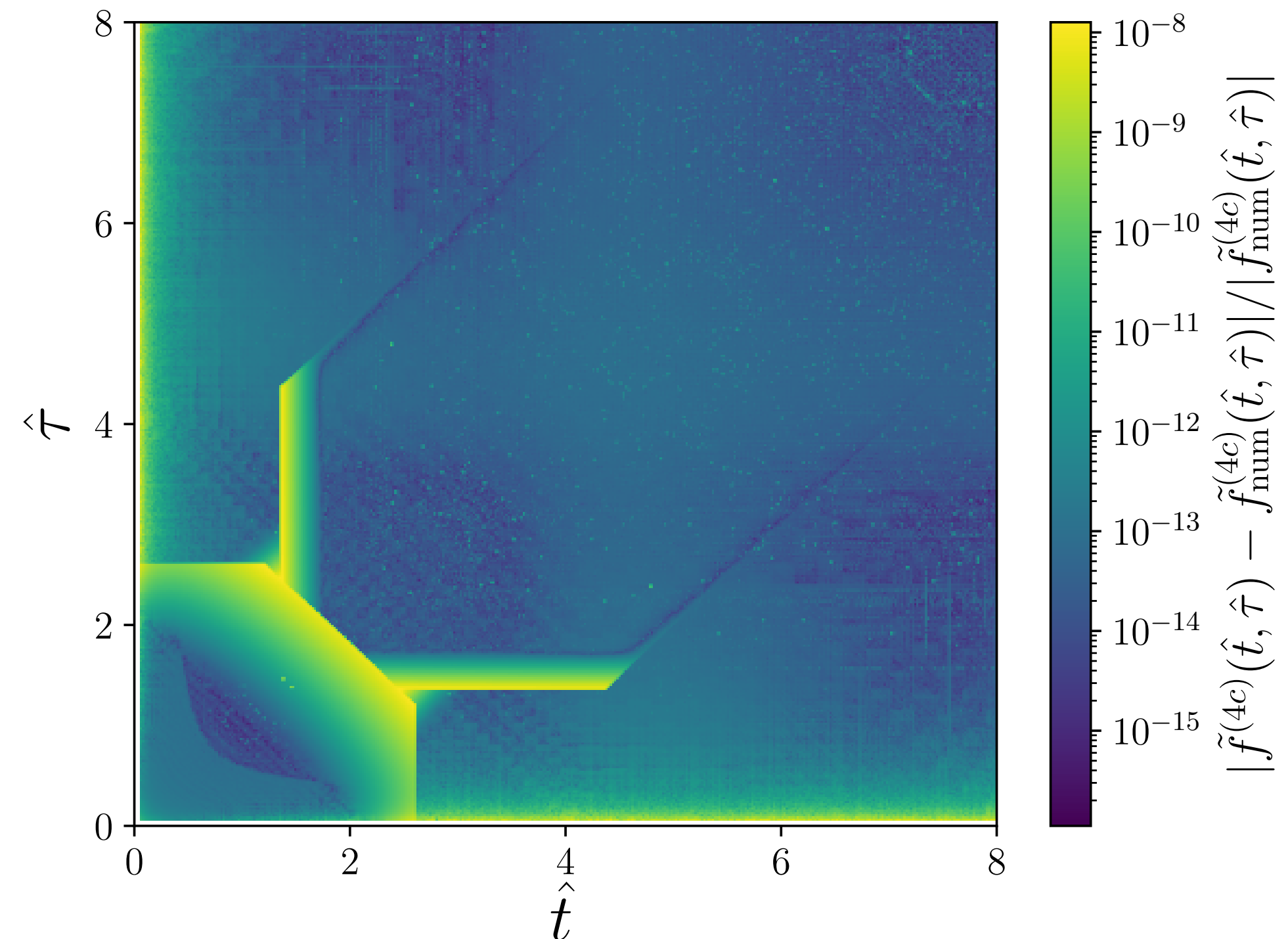
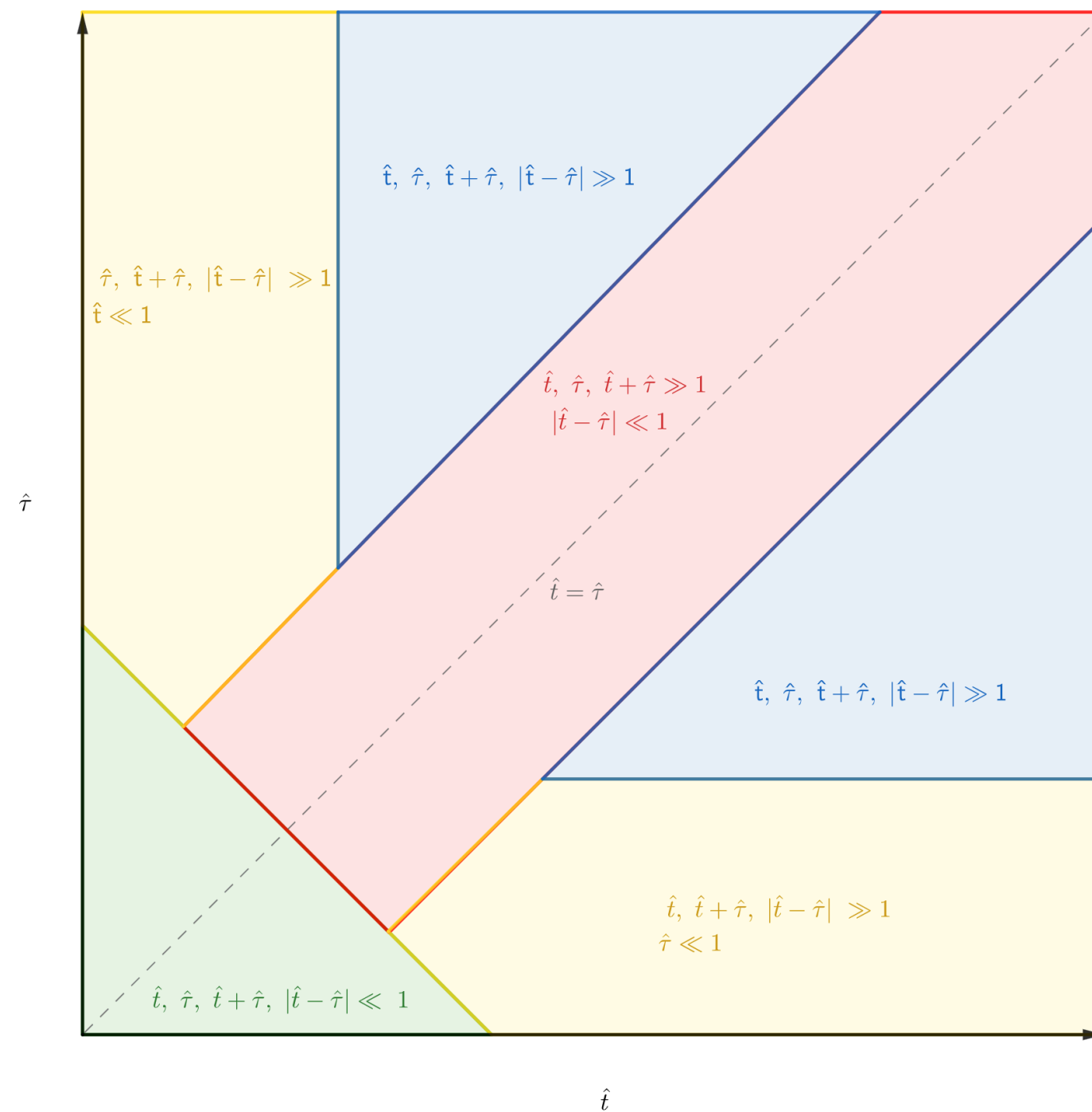
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$$a_{\ell}^{\text{hvp}, (i)} = \left(\frac{\alpha}{\pi}\right)^{2+N_i} \int_0^{\infty} dq^2 f^{(i)}(\hat{q}^2) (\bar{\Pi}(q^2))^{m_i}$$

vs

$$a_{\ell}^{\text{hvp}, (i)} = \left(\frac{\alpha}{\pi}\right)^{2+N_i} \int dt_1 \dots dt_{m_i} \tilde{f}^{(i)}(\hat{t}_1, \dots, \hat{t}_{m_i}) G(t_1) \times \dots \times G(t_{m_i})$$

We observe a **relative error**
 $\lesssim 10^{-10}$ for all observables

On the time-momentum representation of the hadronic vacuum polarization contribution to $g - 2$ at next-to-leading order. **Manuscript in preparation, 2026**

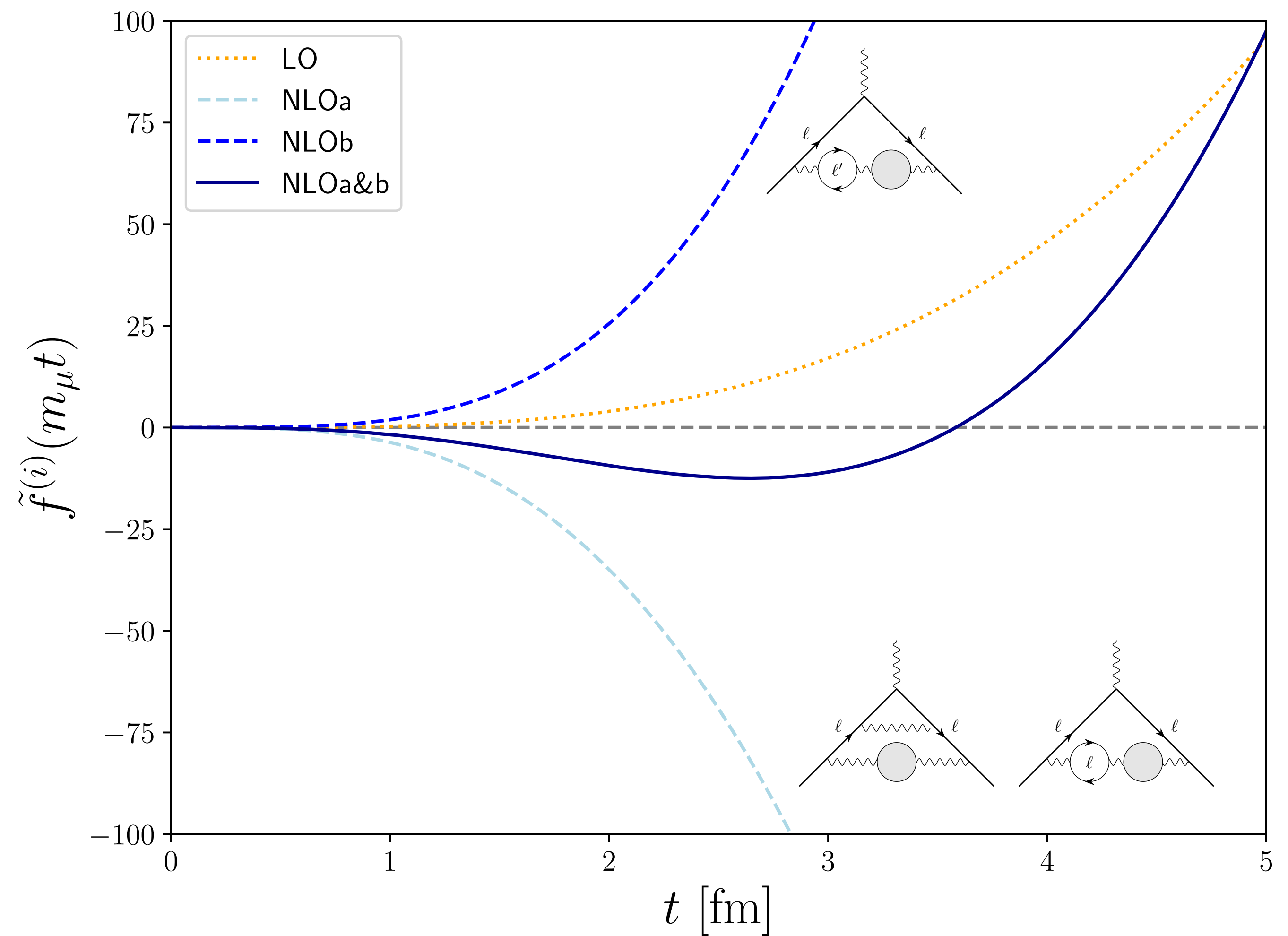
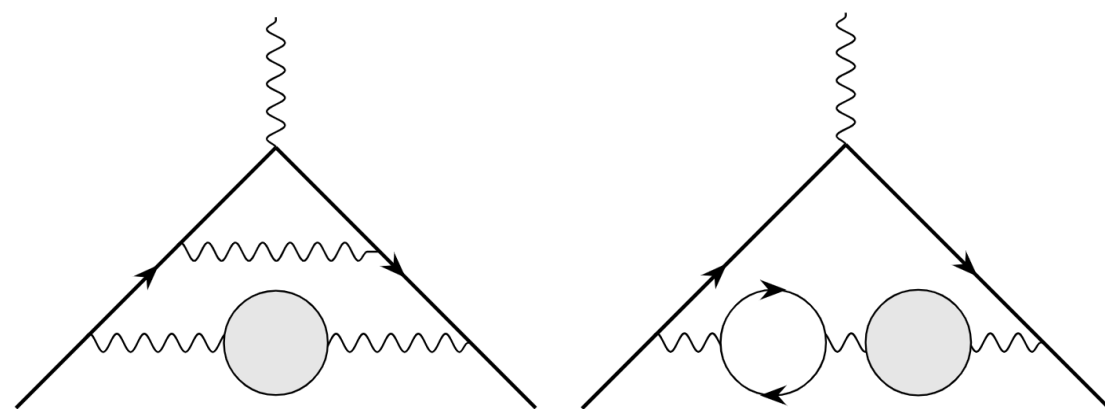
hvpKernels.jl

Long distance cancellation for $\ell = \mu$

NLO HVP

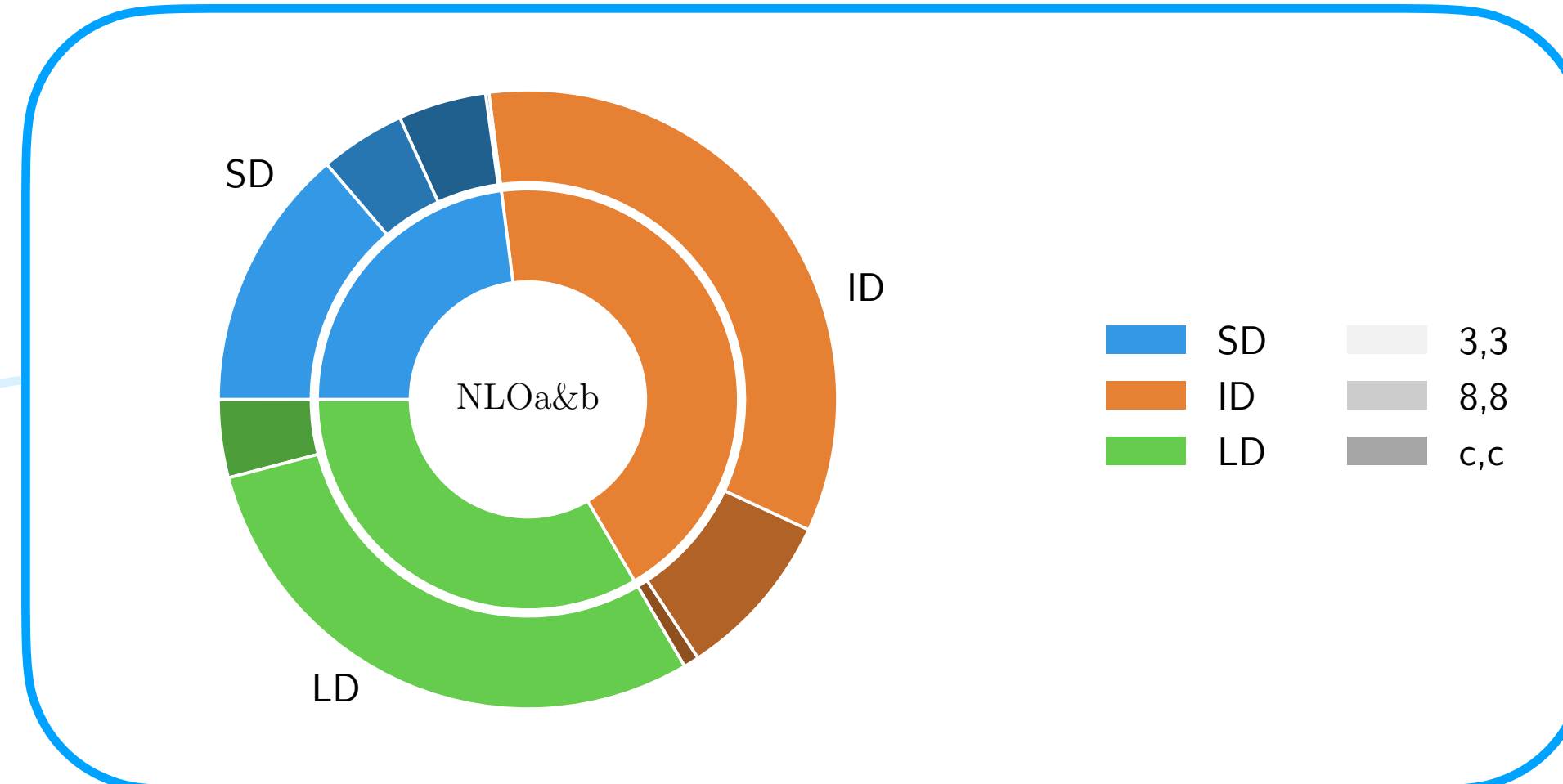
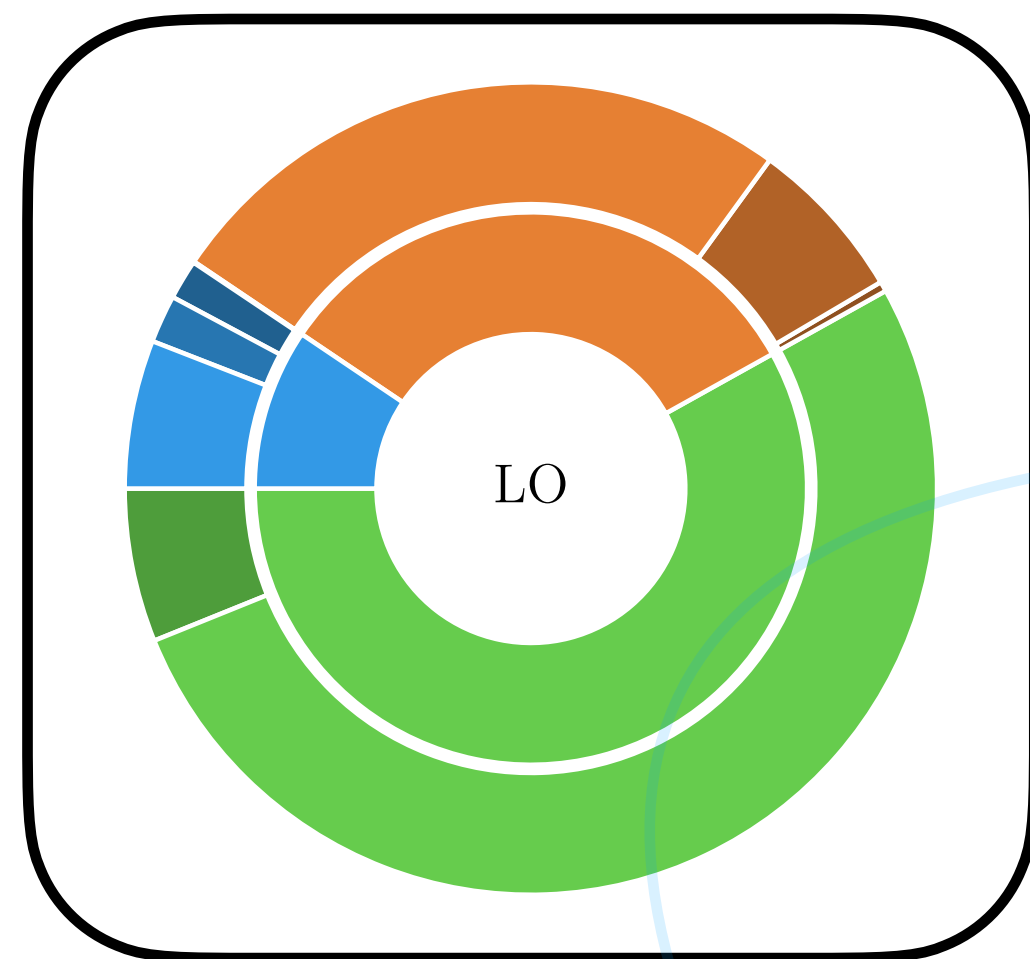
We set $m_\ell = m_\mu$

- NLOa and NLOb have opposite signs
- Their addition exactly cancels around $t \sim 3.6$ fm
- **NLOa&b** $\equiv$ NLOa + NLOb exploits this cancellation

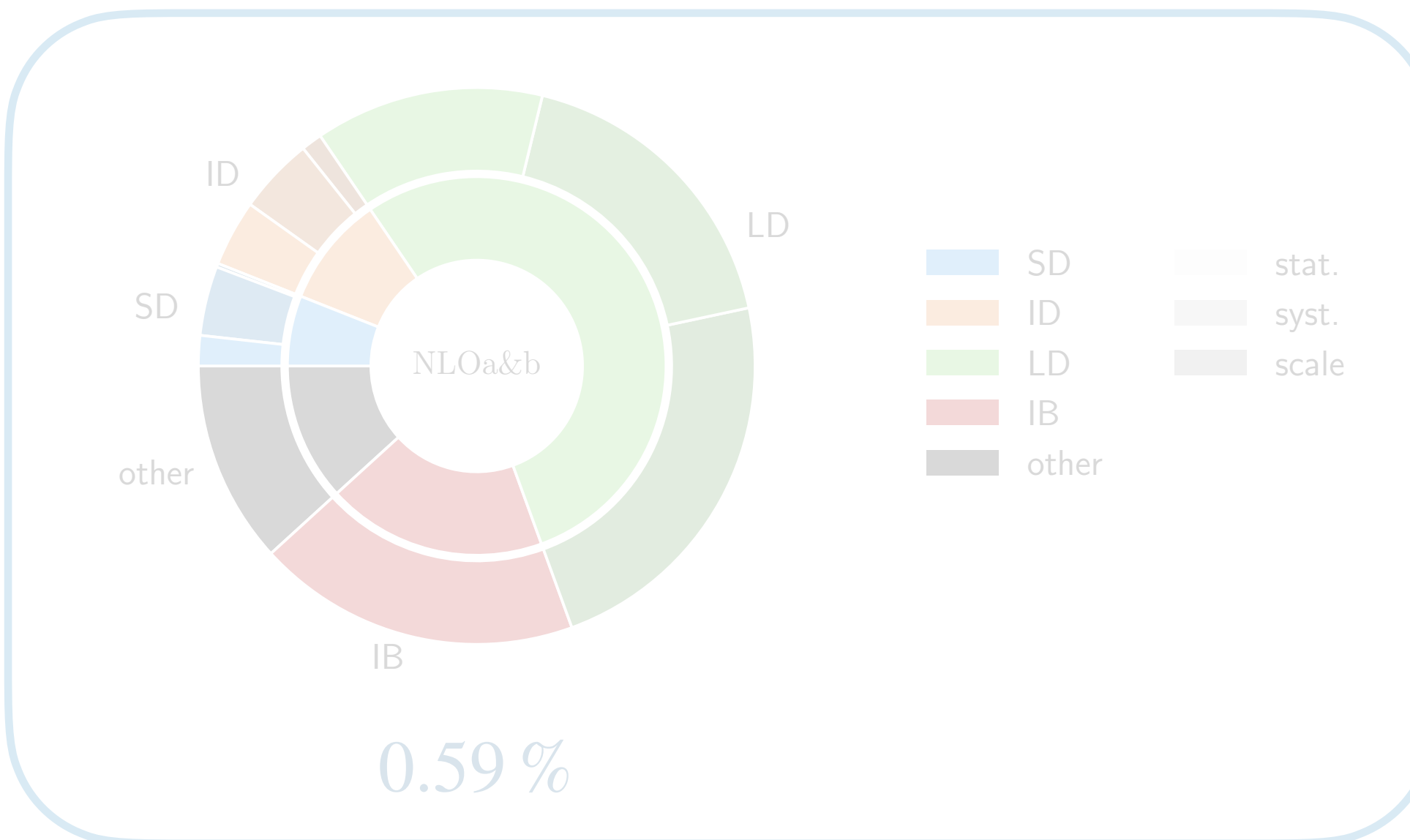


Reduction of uncertainty

NLO HVP



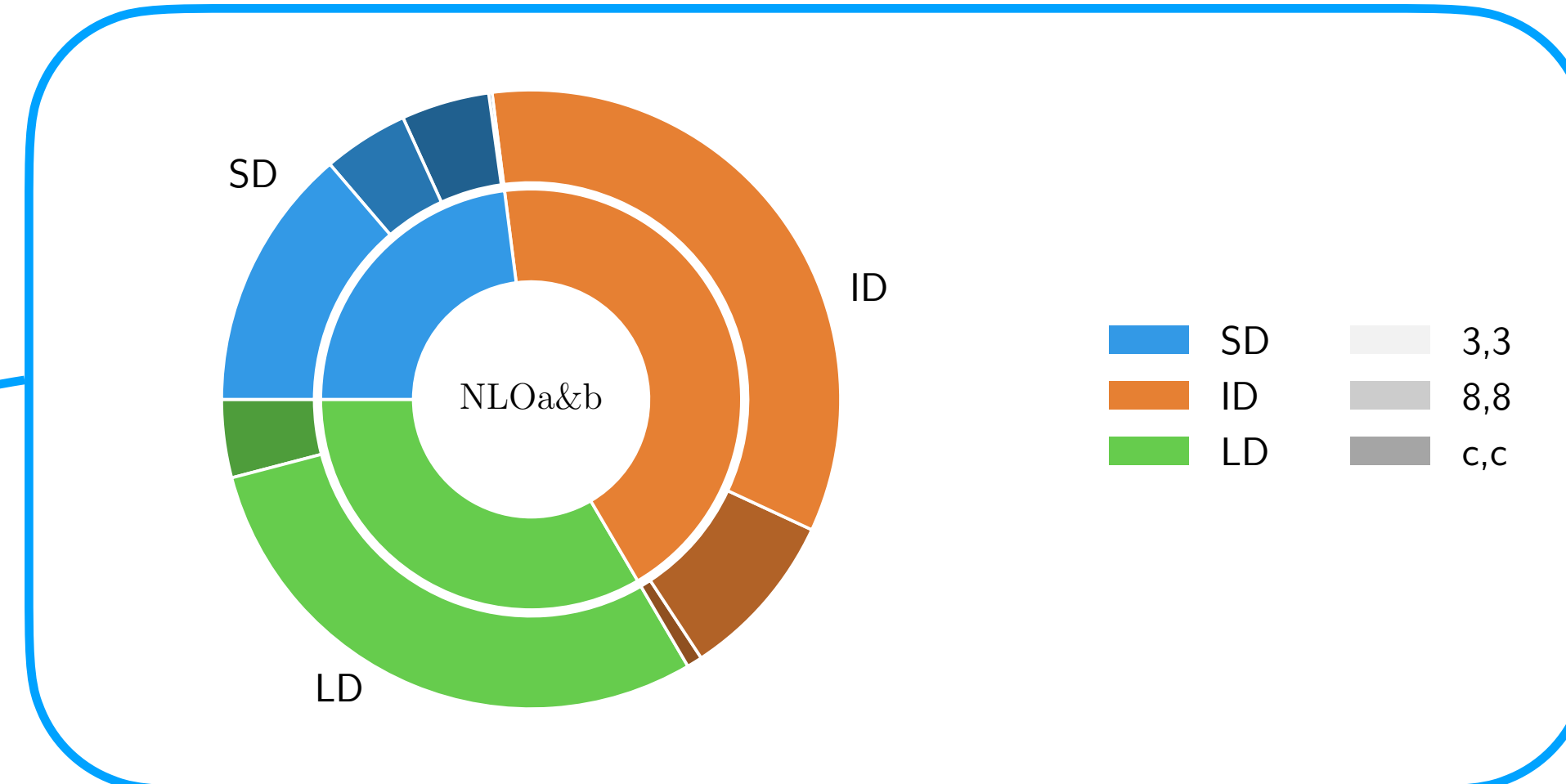
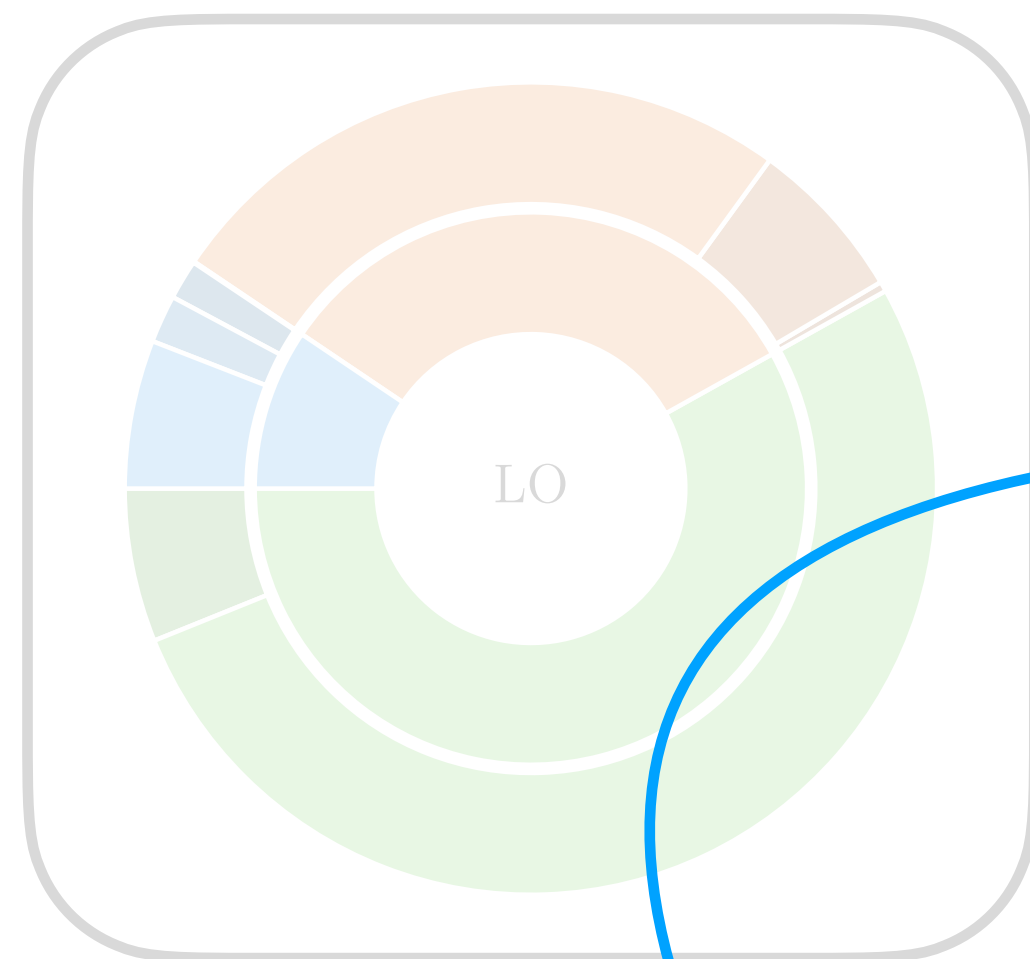
← Contribution to the Central Value



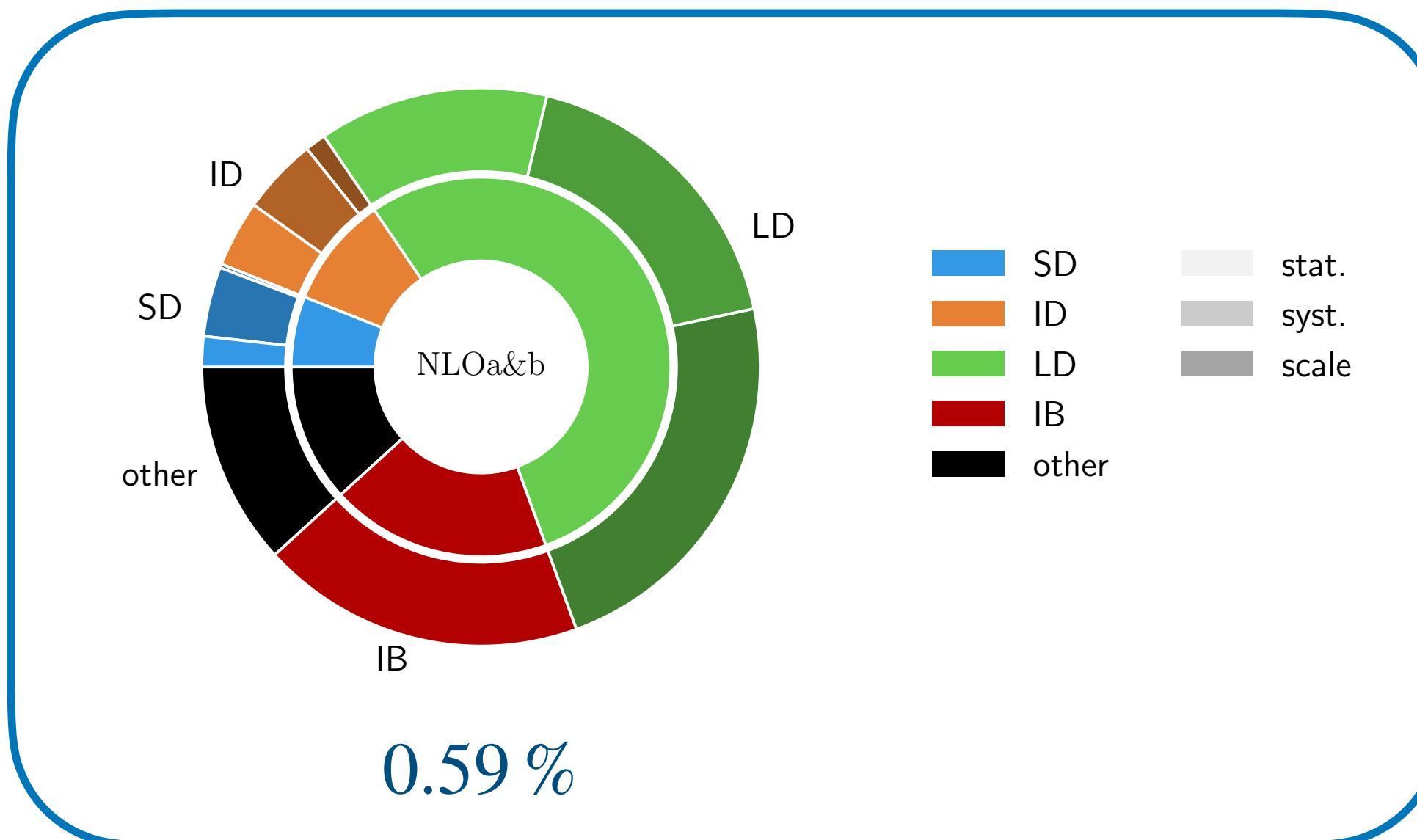
← Contribution to the Final Error

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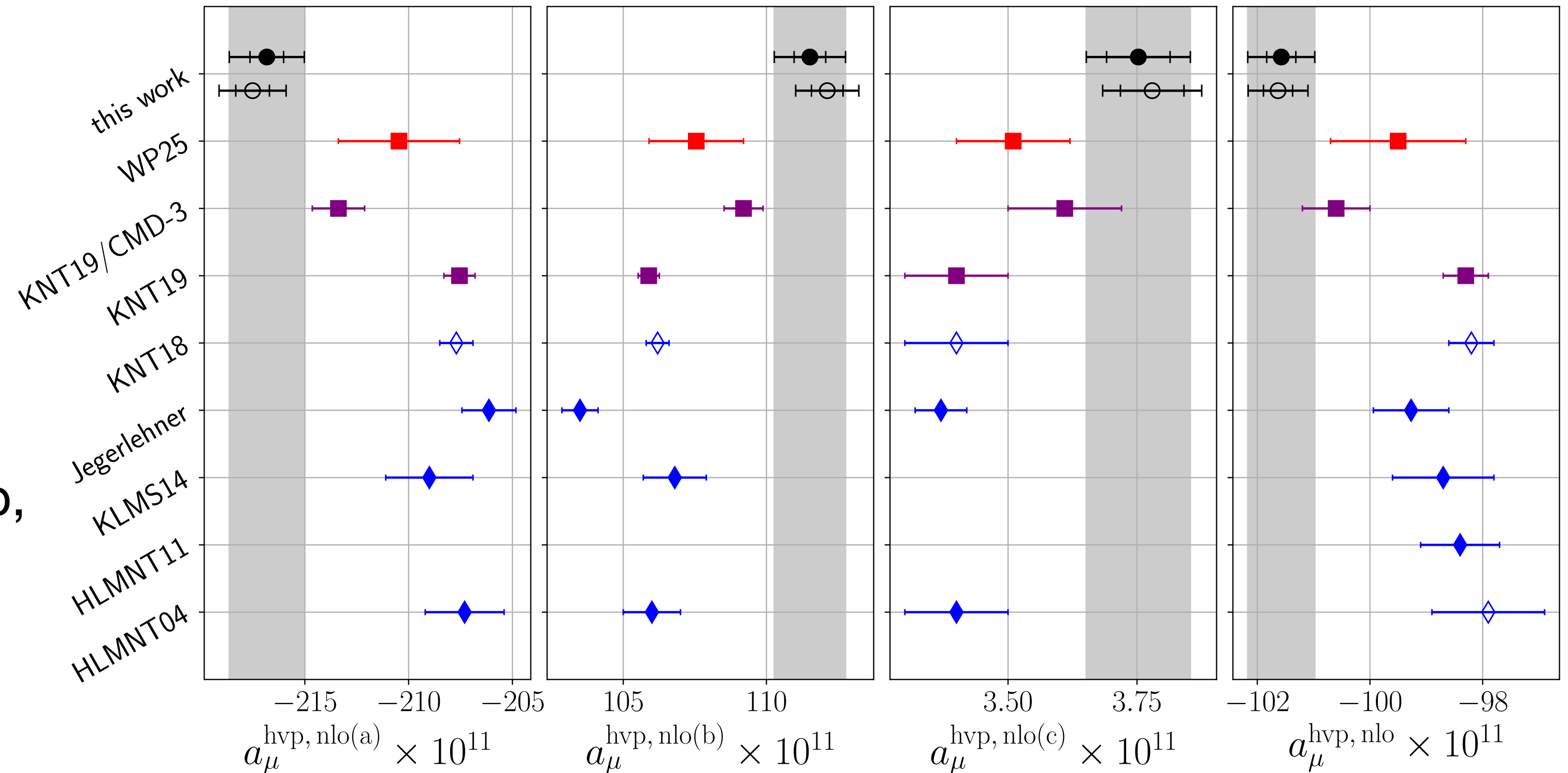


← Contribution to the Central Value



← Contribution to the Final Error

- The size of NLO HVP is similar to the HLbL but with the opposite sign
- We have achieved a 0.6 % precision, improving on WP25
- Large cancelation in the Isospin Breaking corrections in NLOa+NLOb, *very reliable QCD estimate*



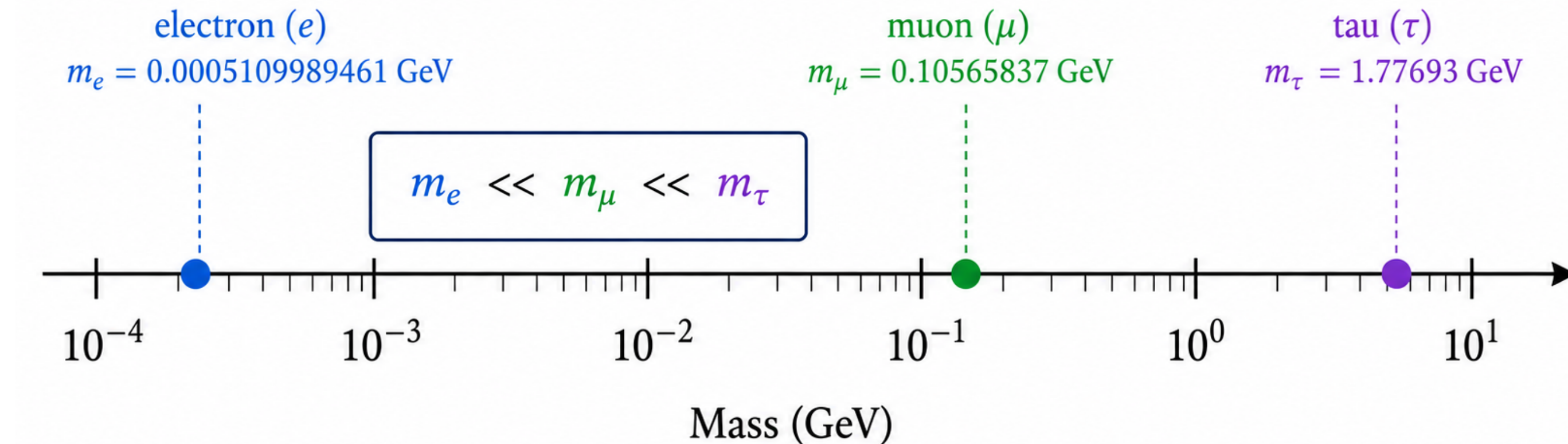
Higher-order hadronic vacuum polarization contribution to the muon $g - 2$ from lattice QCD. 2026. A. Beltran et al.

$$a_{\mu}^{\text{hvp, nlo}} = -101.57(26)_{\text{stat}}(54)_{\text{syst}}[60] \times 10^{-11}$$

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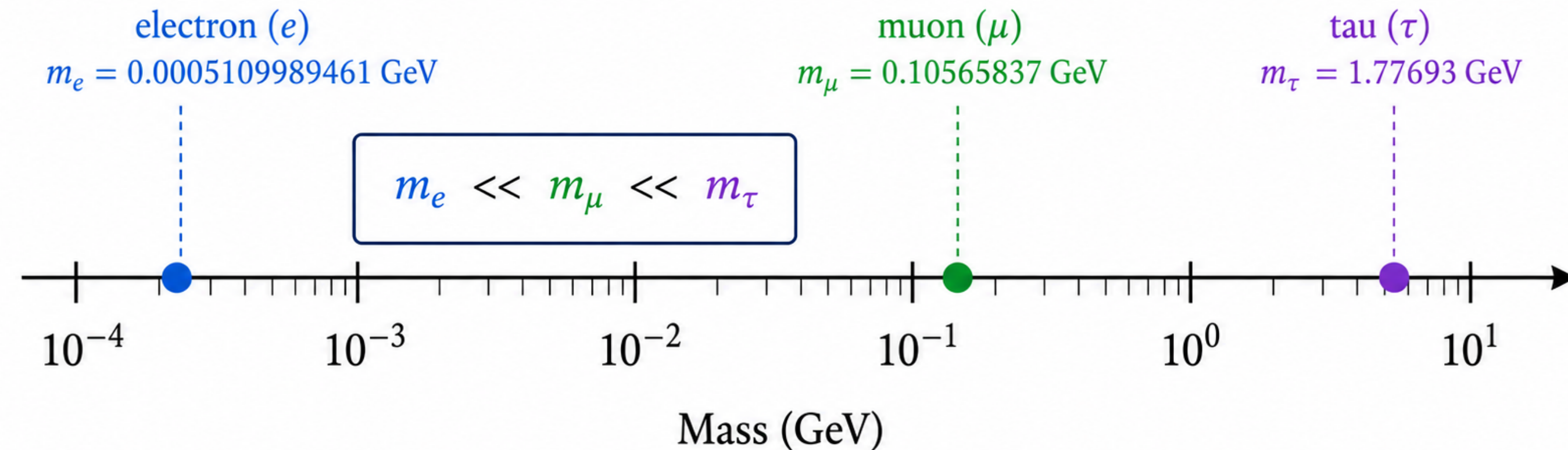
Lepton flavor hierarchy

$g_\ell - 2$



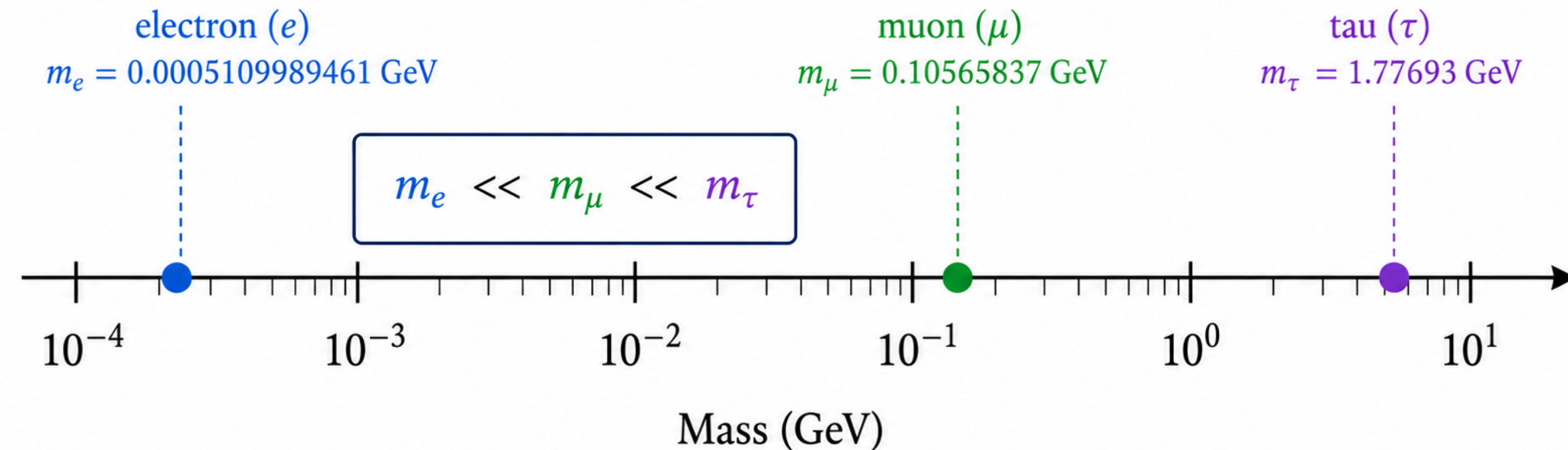
- $a_\ell^{\text{HVP}} \sim \frac{m_l^2}{\Lambda^2}$ for small $m_\ell \rightarrow$ HVP contributions remain small for a_e (and a_μ)
- The smallness of m_e (m_μ) pushes the bulk of the integrand towards *the long distances* $\rightarrow$ **StN problem**
- The (relative) stability of e (μ) allows for an **accurate experimental determination** of a_e (a_μ)

- m_τ pushes the bulk of the integrand towards *the short distances* $\rightarrow$ we can estimate a_τ^{HVP} to great precision
- Due to the large decay rate of τ we do not have precise enough experimental results to probe a_τ^{HVP}



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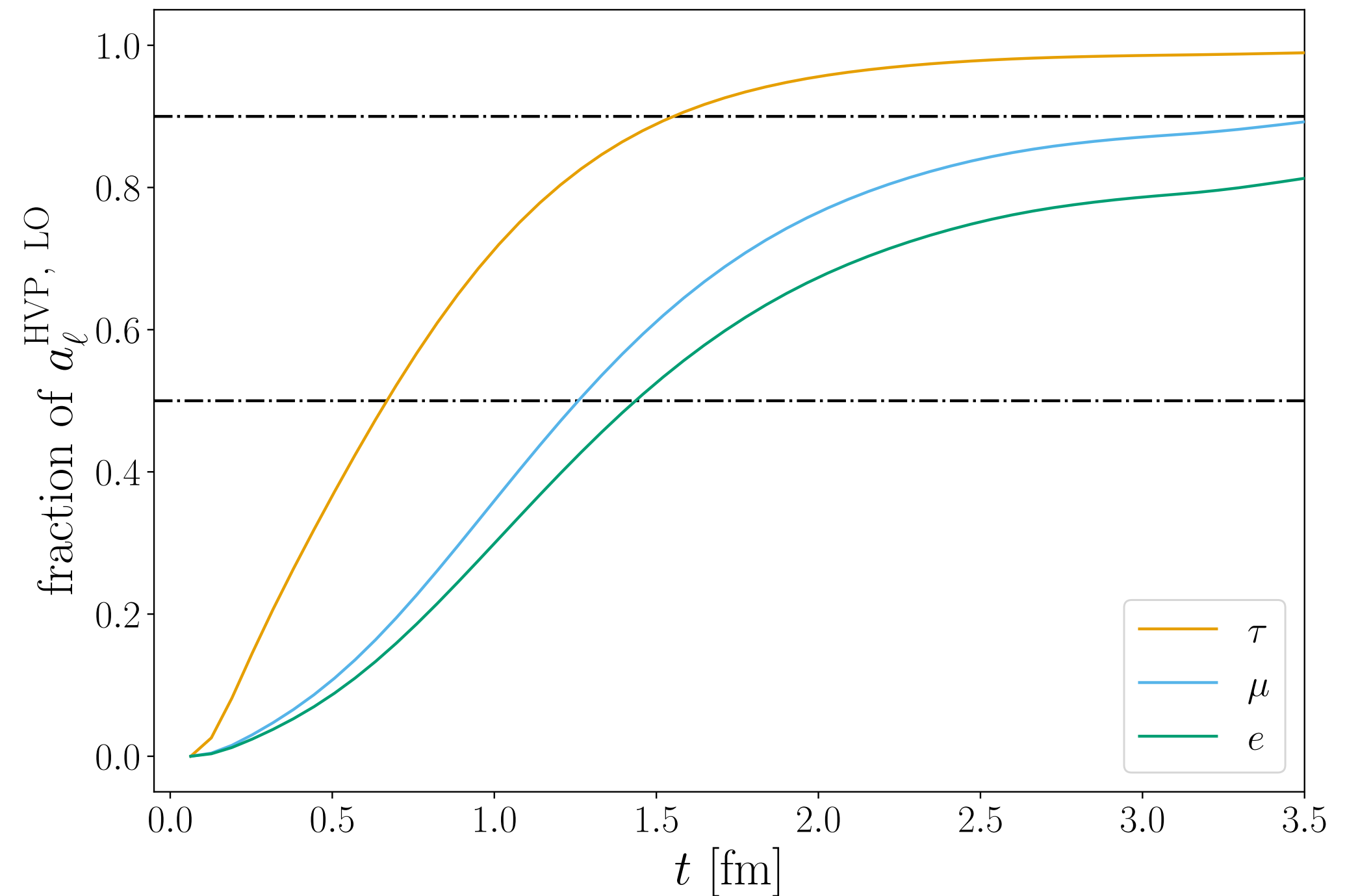
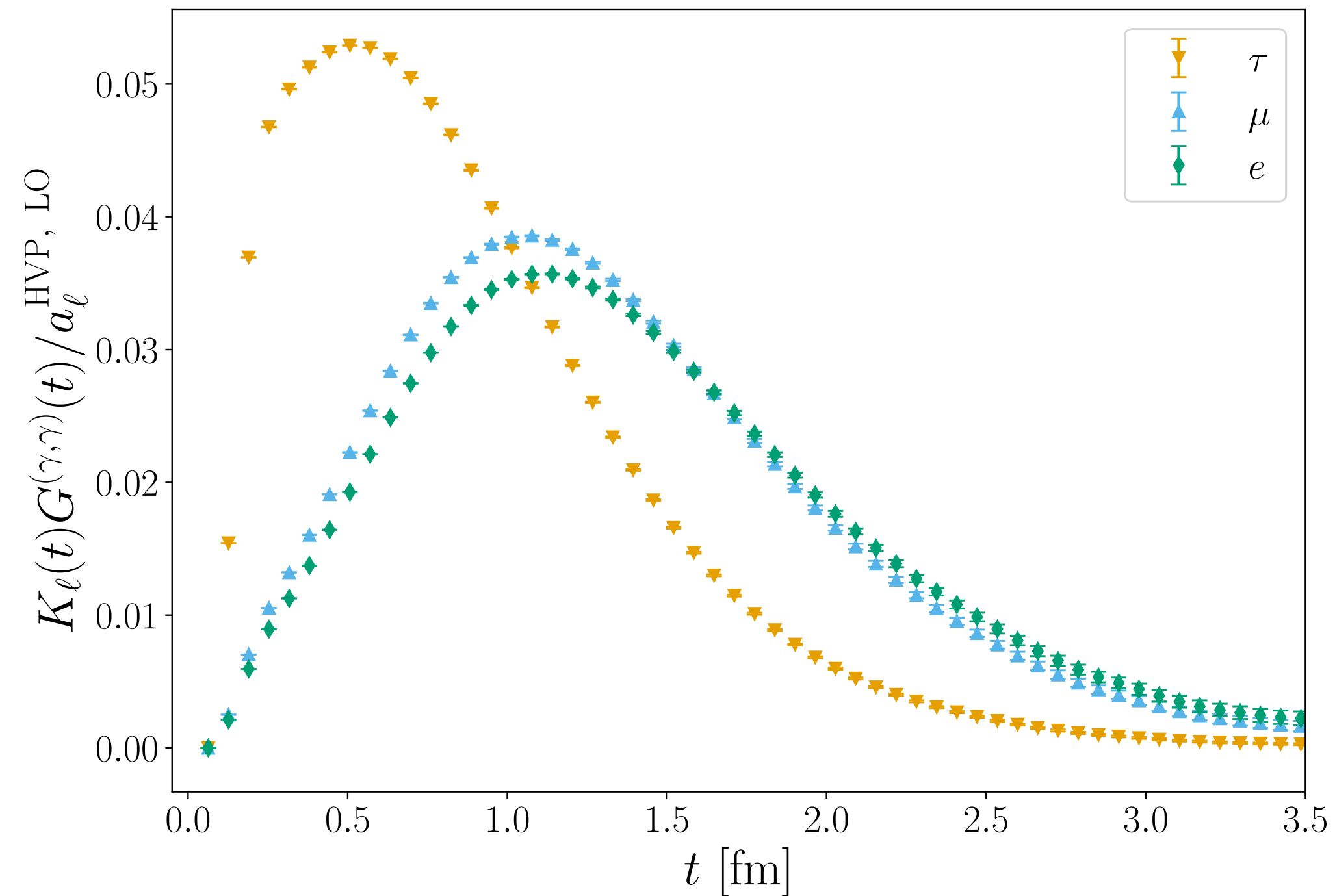
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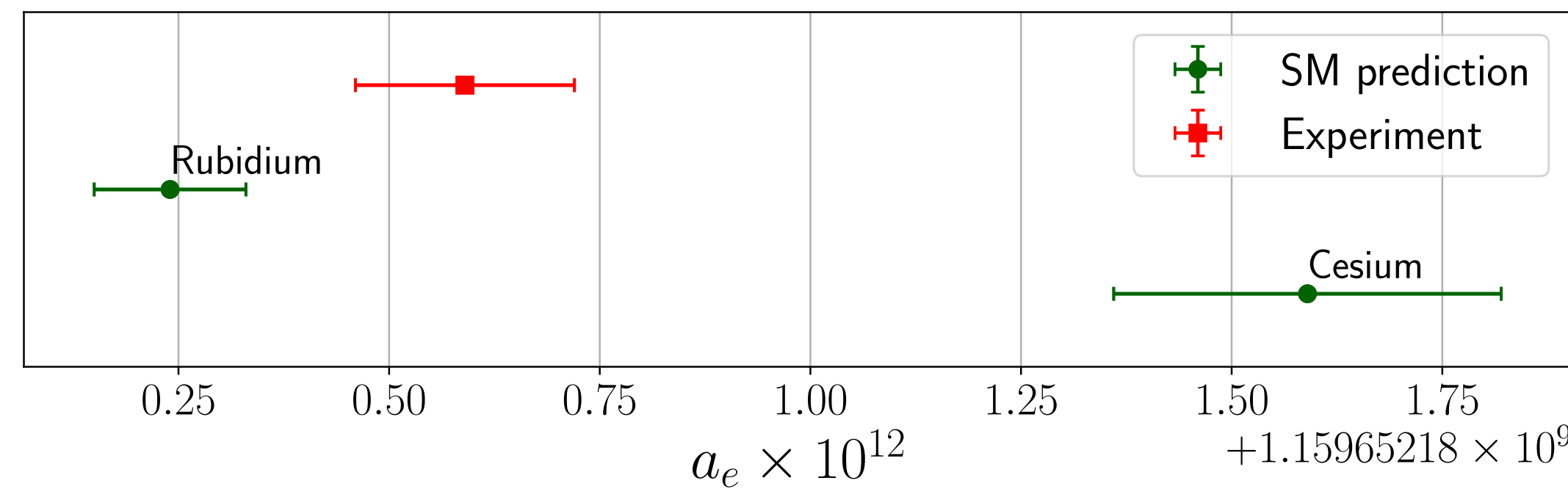
Lepton universality at the LO level

$g_\ell - 2$

$$a_\ell^{\text{hvp, lo}} = \left(\frac{\alpha}{\pi} \right)^2 \int_0^\infty dt \tilde{f}^{(2)}(m_\ell t) G(t)$$

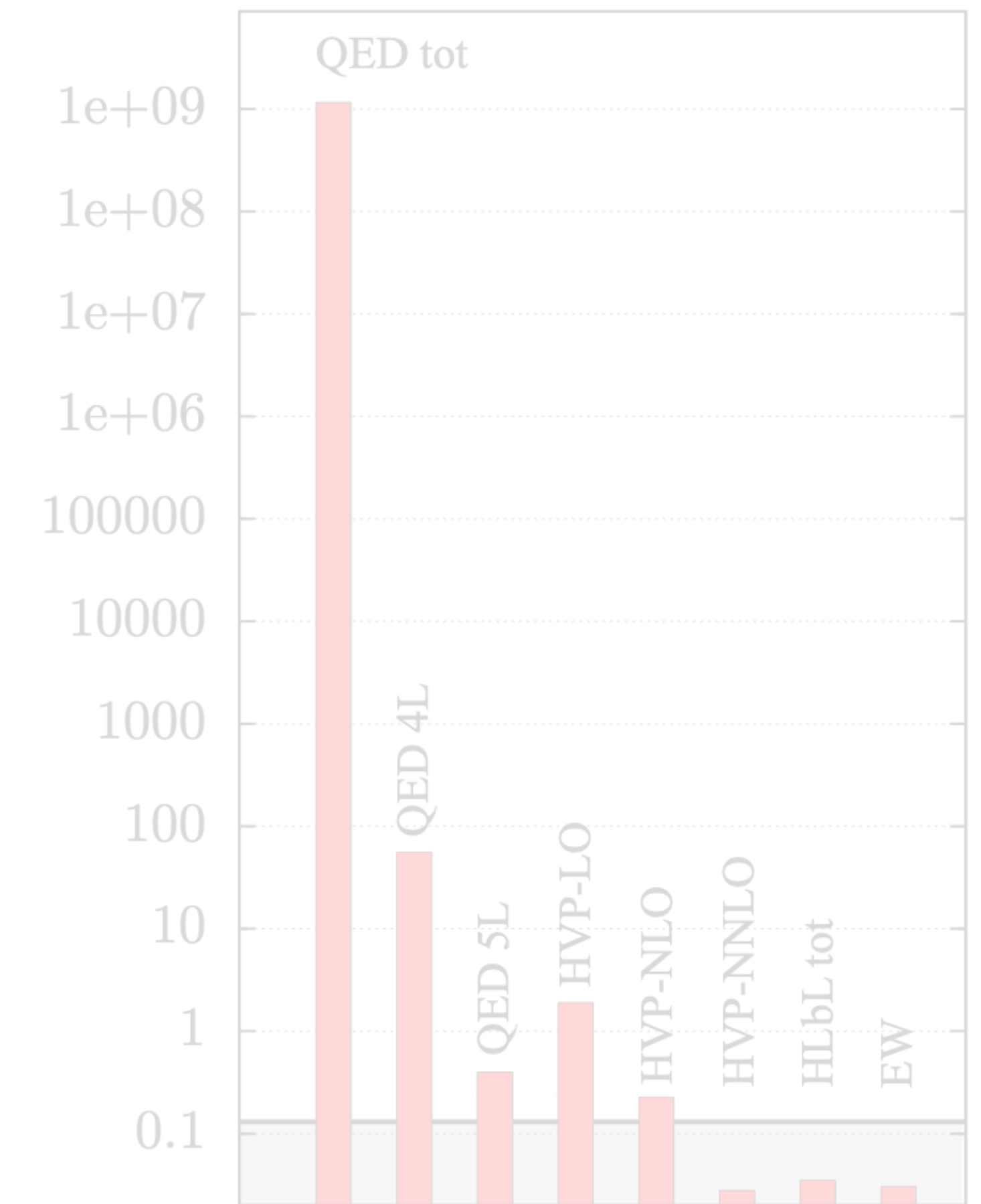


- Current uncertainty on a_e is dominated by the **Cs/Rb discrepancy** in determinations of α (*atom interferometry*)



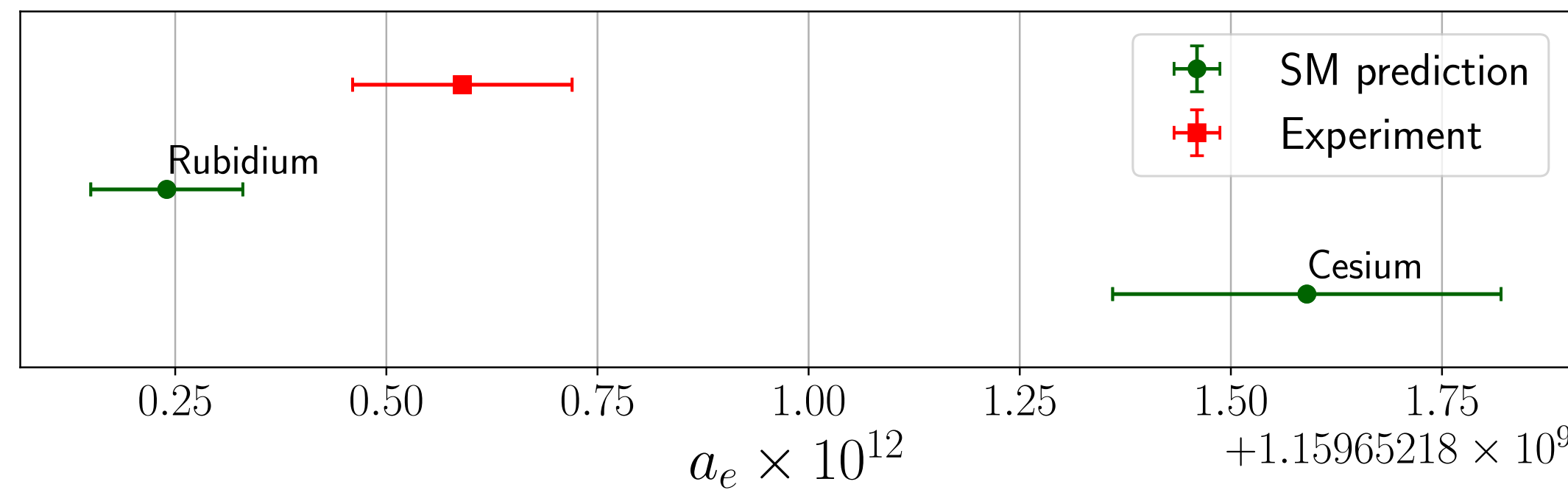
- Current error of a_e^{exp} already **sensitive to the size** of $a_e^{\text{hvp, lo}}$ and with prospects of improving
- a_e^{HVP} suffers of the *usual discrepancies* in the **data-driven approach**
- Once α is resolved, **an unambiguous determination** of $a_e^{\text{hvp, lo}}$ is as pressing as a complete 6th-order QED computation

[H. Wittig 2025]



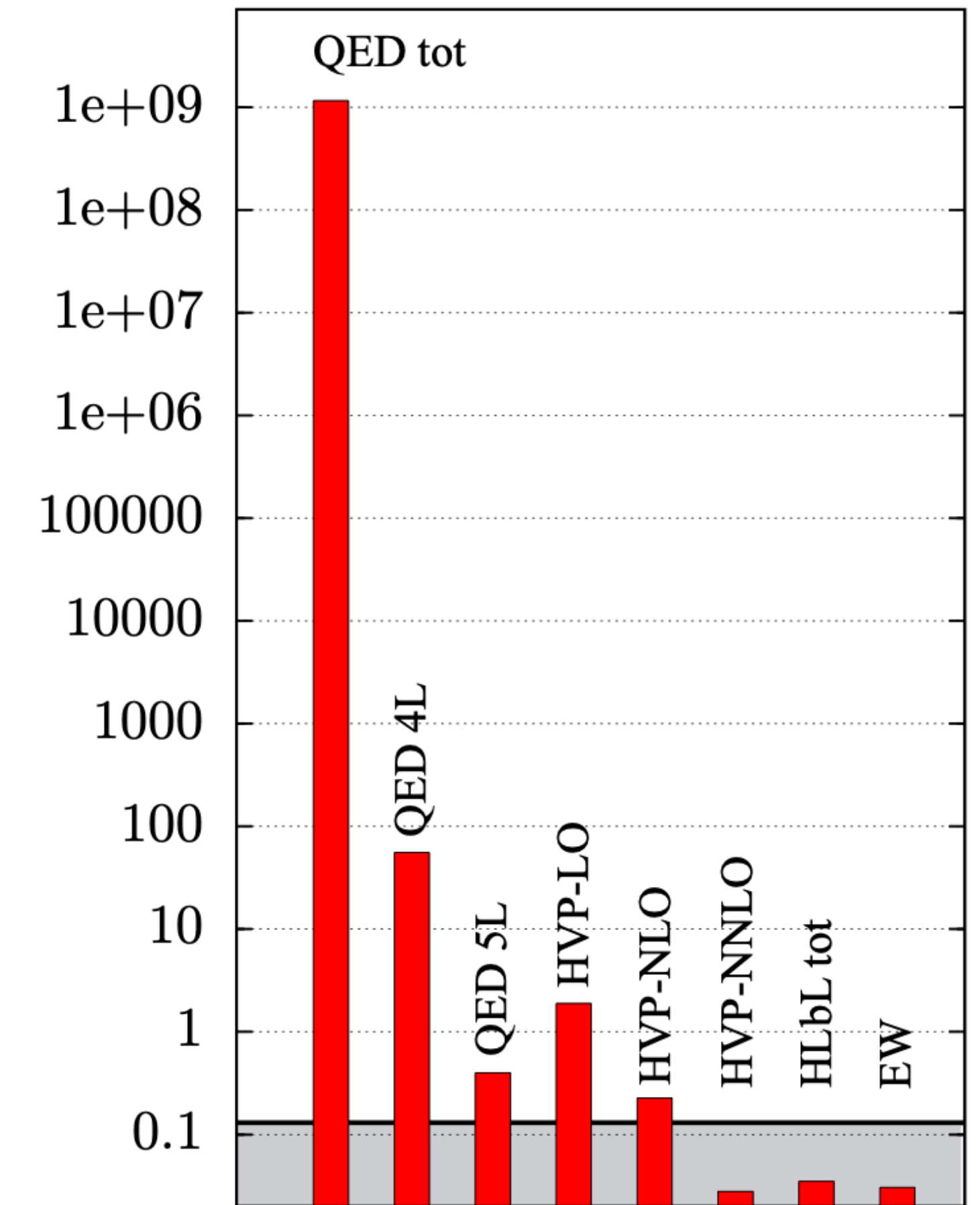
Relative contribution to $a_e \times 10^{12}$ of different sectors of the SM

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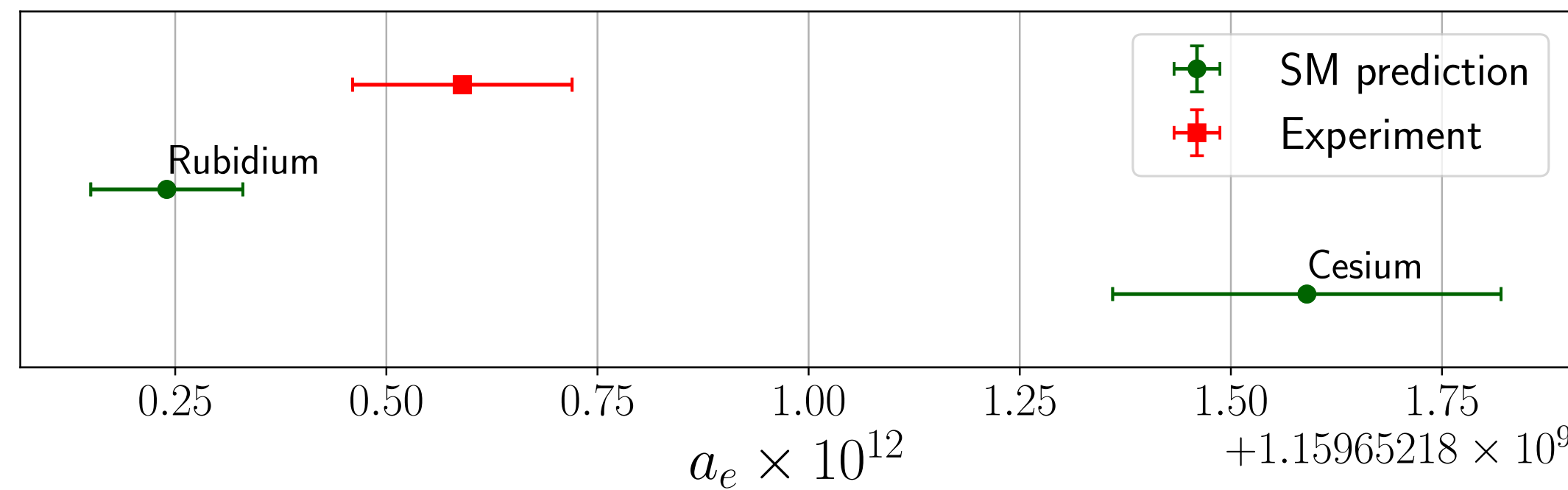
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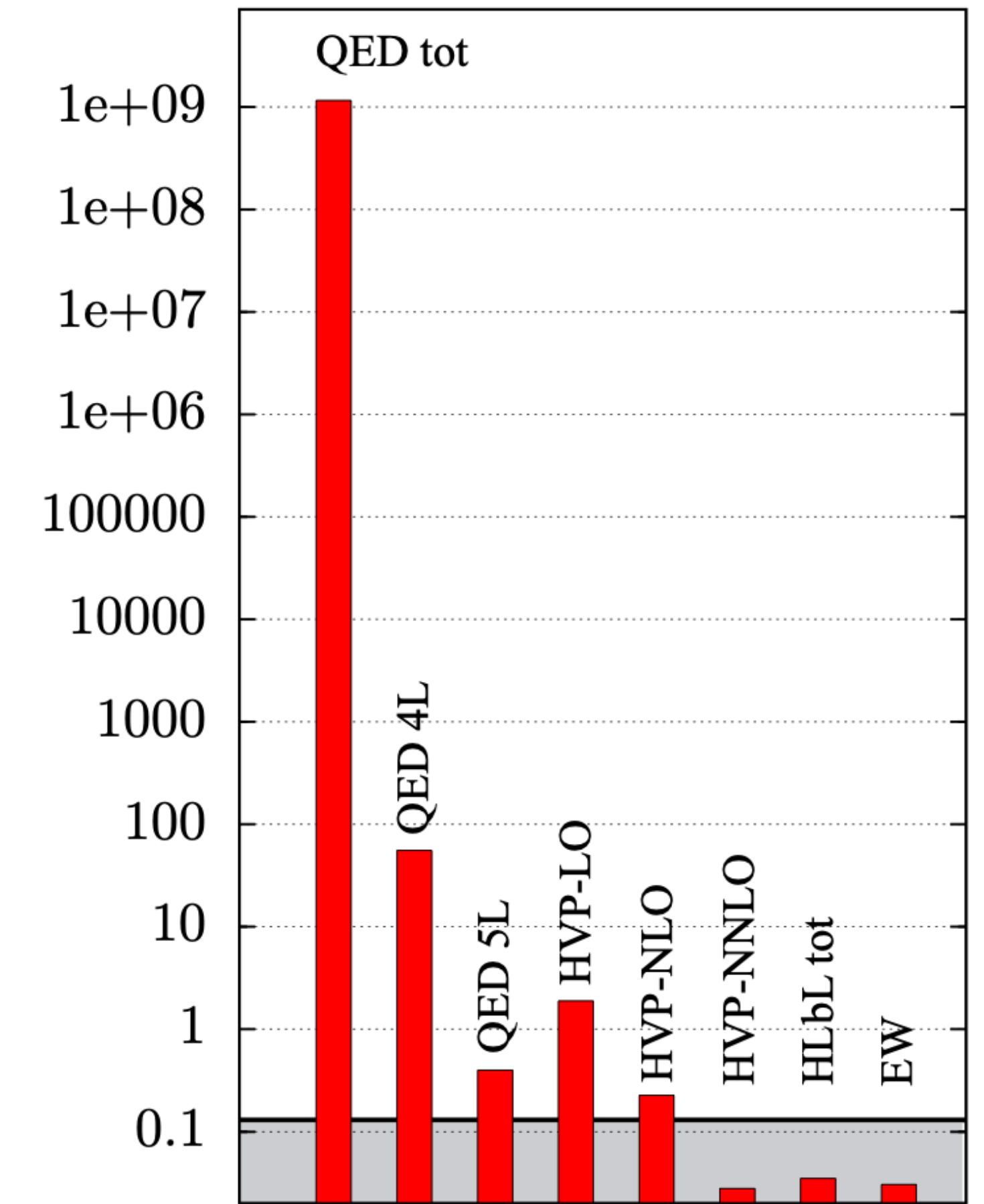
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[H. Wittig 2025]



Relative contribution to $a_e \times 10^{12}$ of different sectors of the SM

$$a_{e-\mu} \equiv a_e - \left(\frac{m_e}{m_\mu} \right)^2 a_\mu$$

[S. Li et al. 2026]

See S. Li talk: Friday 12:15 - 12:35
(V. Pascalutsa: Tuesday 9:30 - 9:50)

- This observable provides a cancelation of high-energetic effects ($\Lambda \gg m_e, m_\mu$)

- At the kernel level, there is an **exact cancellation** of the $O(t^4)$ terms

$$a_e^{\text{hvp, lo}} - \left(\frac{m_e}{m_\mu} \right)^2 a_\mu^{\text{hvp, lo}} \longleftrightarrow \tilde{f}^{(2)}(m_e t) - \left(\frac{m_e}{m_\mu} \right)^2 \tilde{f}^{(2)}(m_\mu t) \stackrel{t \ll 1/m_\mu \ll 1/m_e}{=} \left(O(m_\mu)^4 + O(m_e)^4 \right) t^6$$

$$\tilde{f}^{(2)}(\hat{t}) \stackrel{\hat{t} \ll 1}{\approx} \frac{\pi^2 m_\ell^2}{9} t^4$$

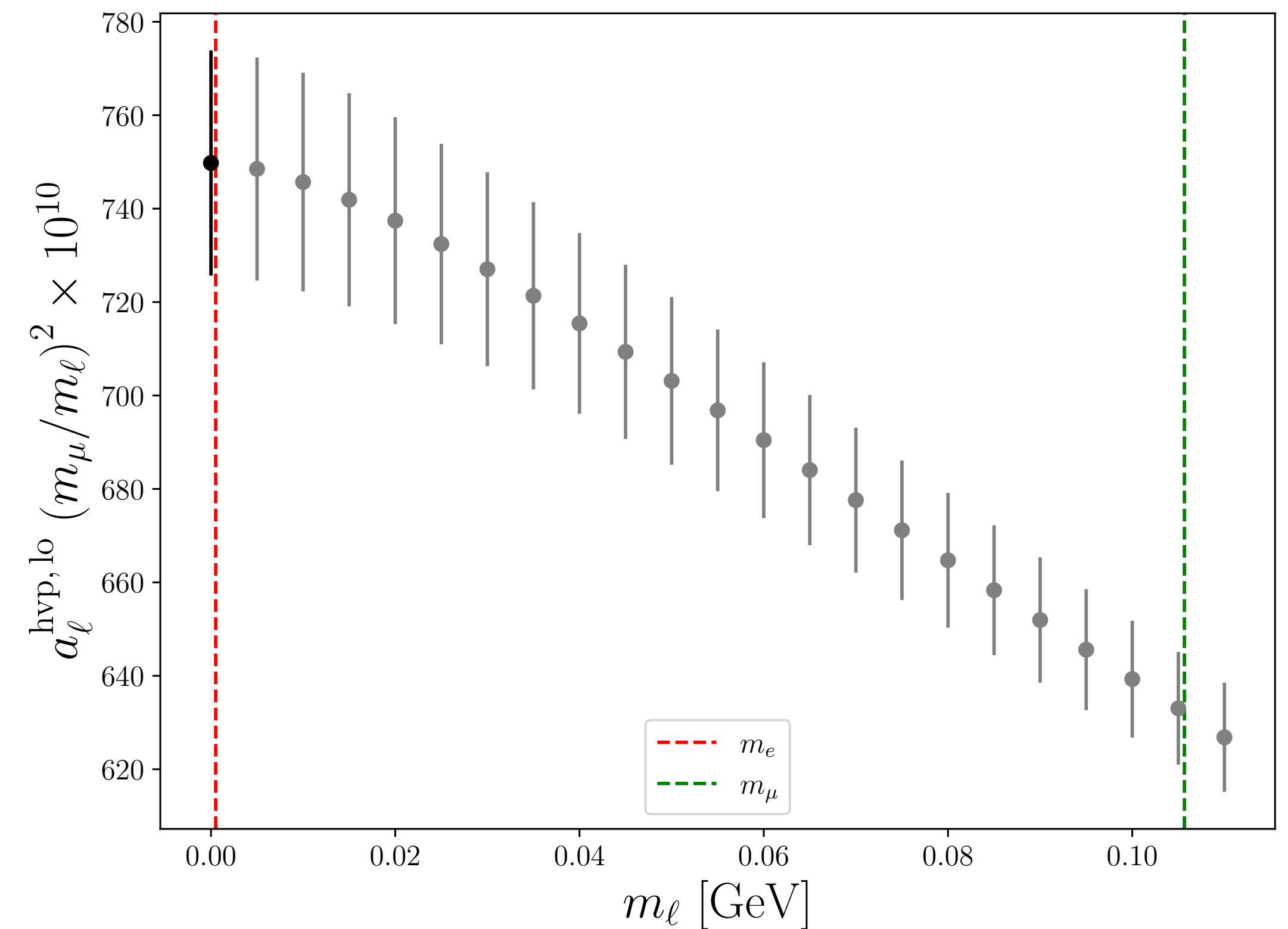
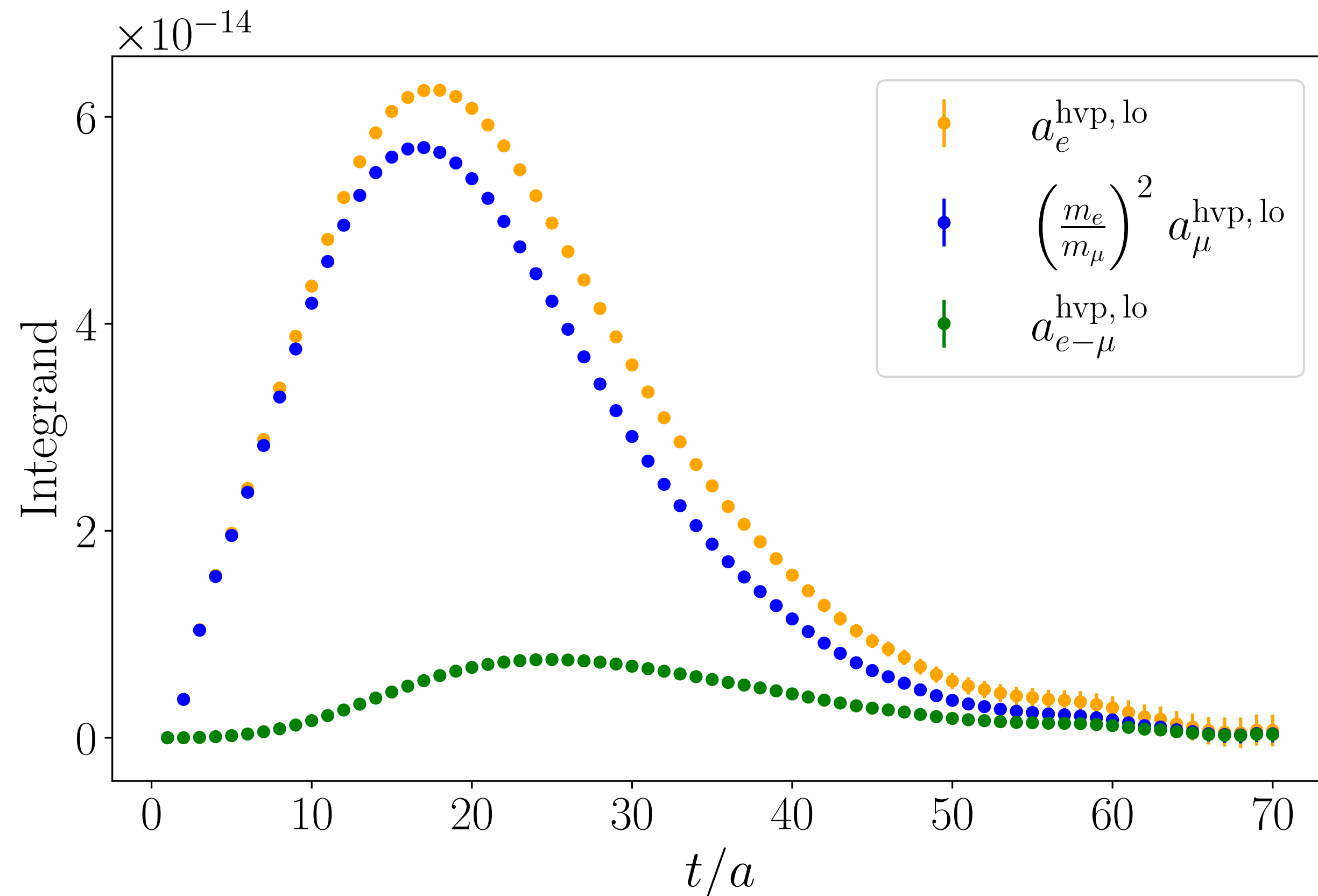
- $a_e^{\text{hvp, lo}}$ is computed as a **shift** from the rescaled $a_\mu^{\text{hvp, lo}}$

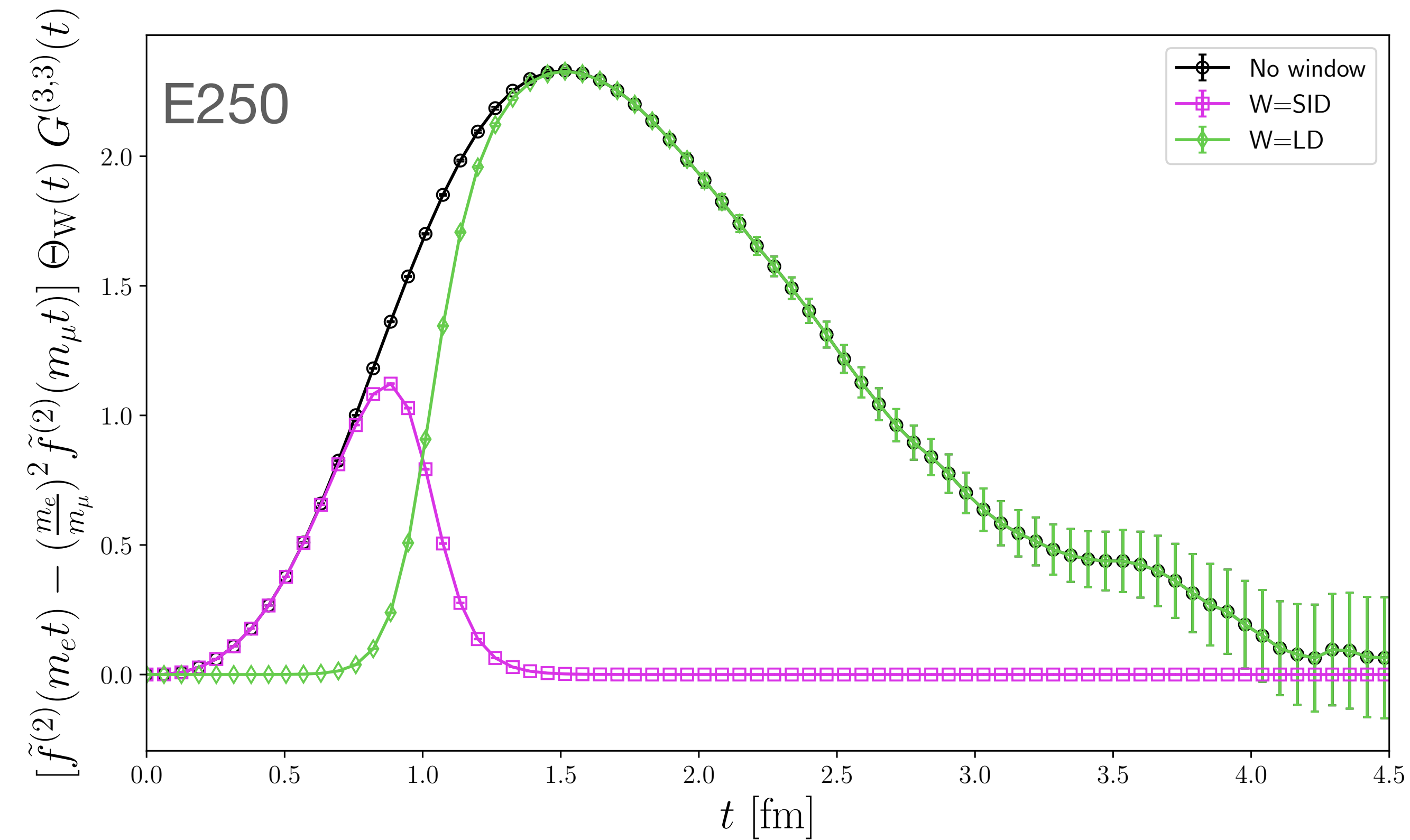
$$a_e^{\text{hvp, lo}} = \left(\frac{m_e}{m_\mu} \right)^2 a_\mu^{\text{hvp, lo}} + a_{e-\mu}^{\text{hvp, lo}}, \quad \frac{a_{e-\mu}^{\text{hvp, lo}}}{a_e^{\text{hvp, lo}}} \sim 15 \%$$

Rescaling from $a_\mu^{\text{hvp, lo}}$

Electron $g - 2$

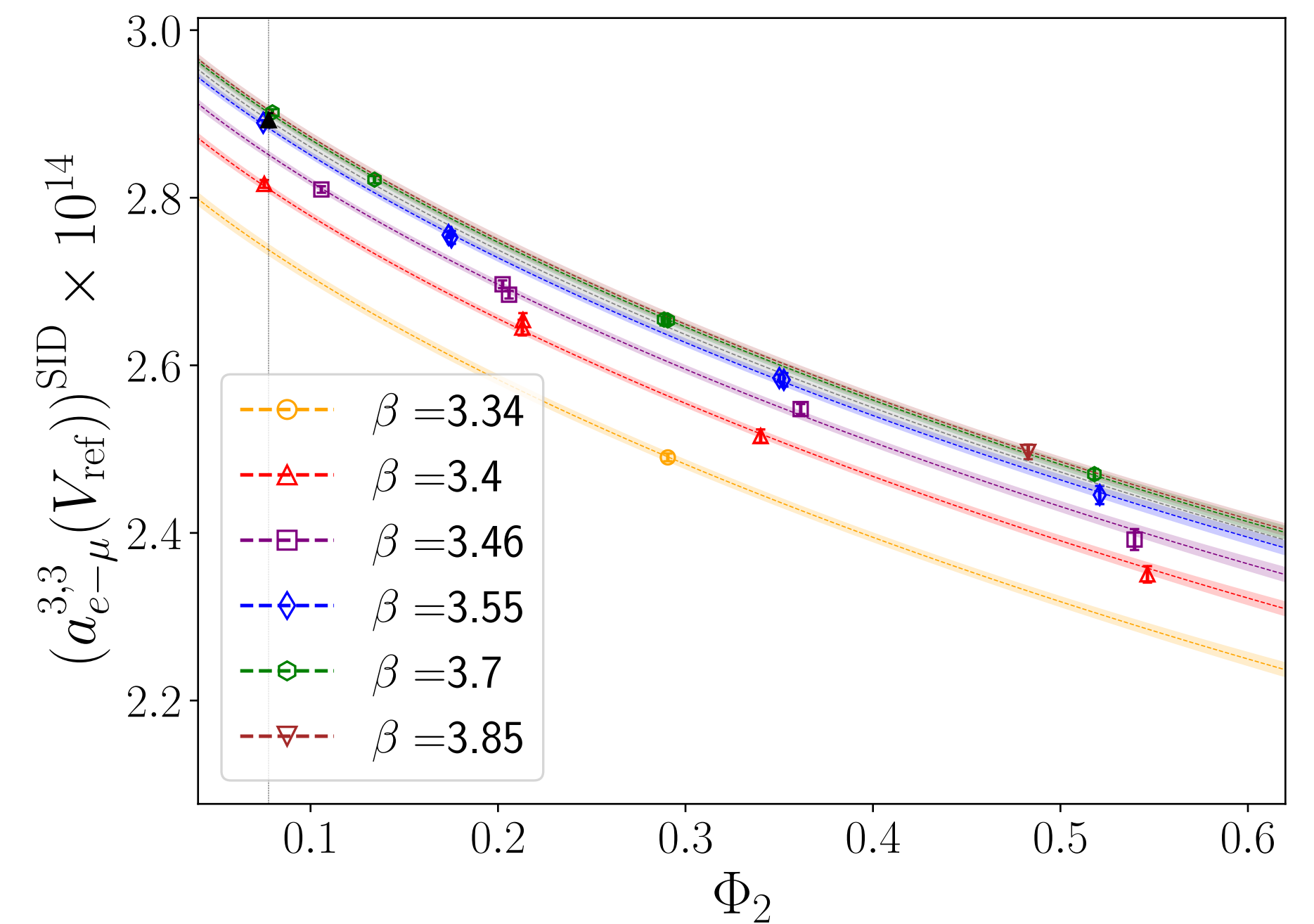
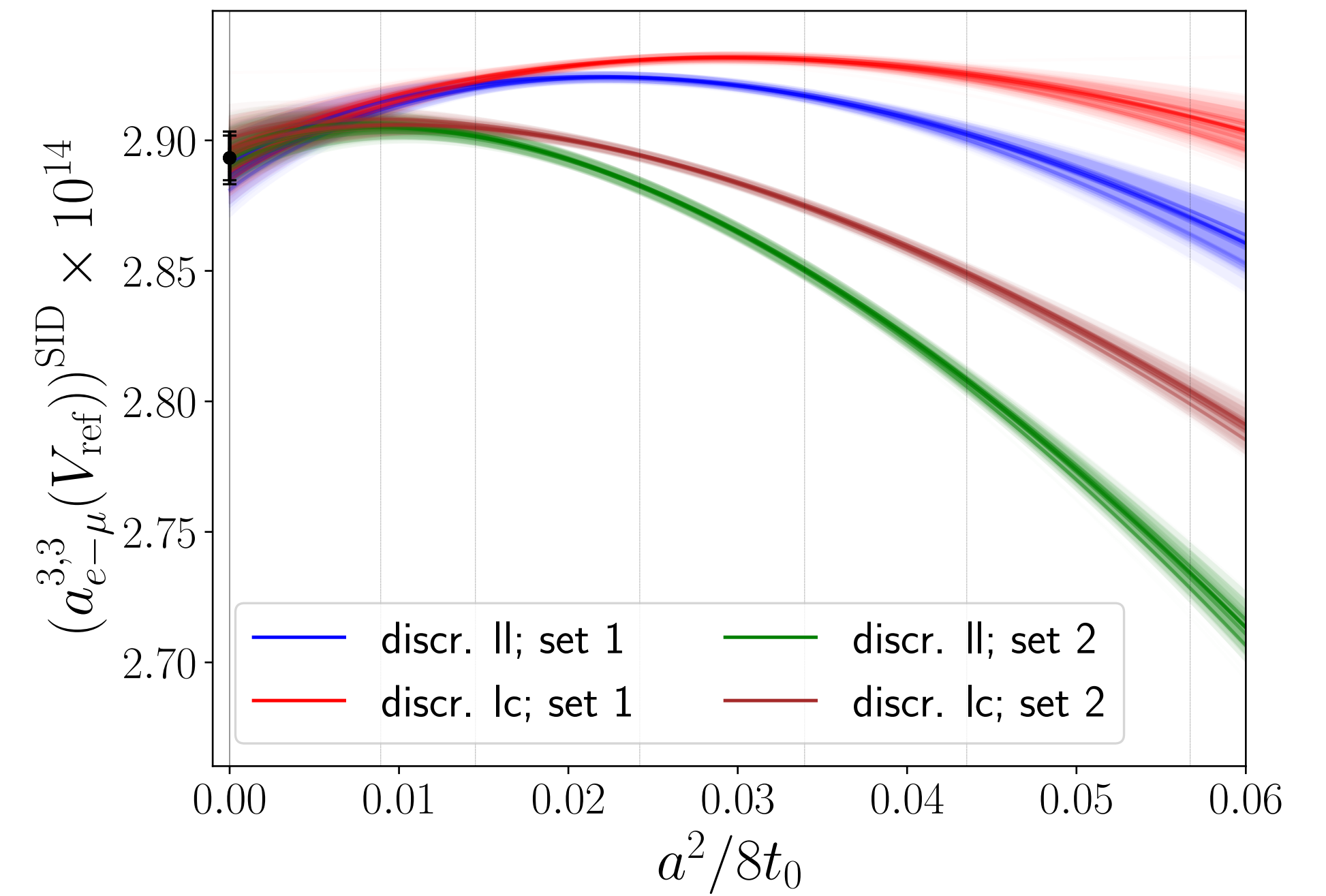
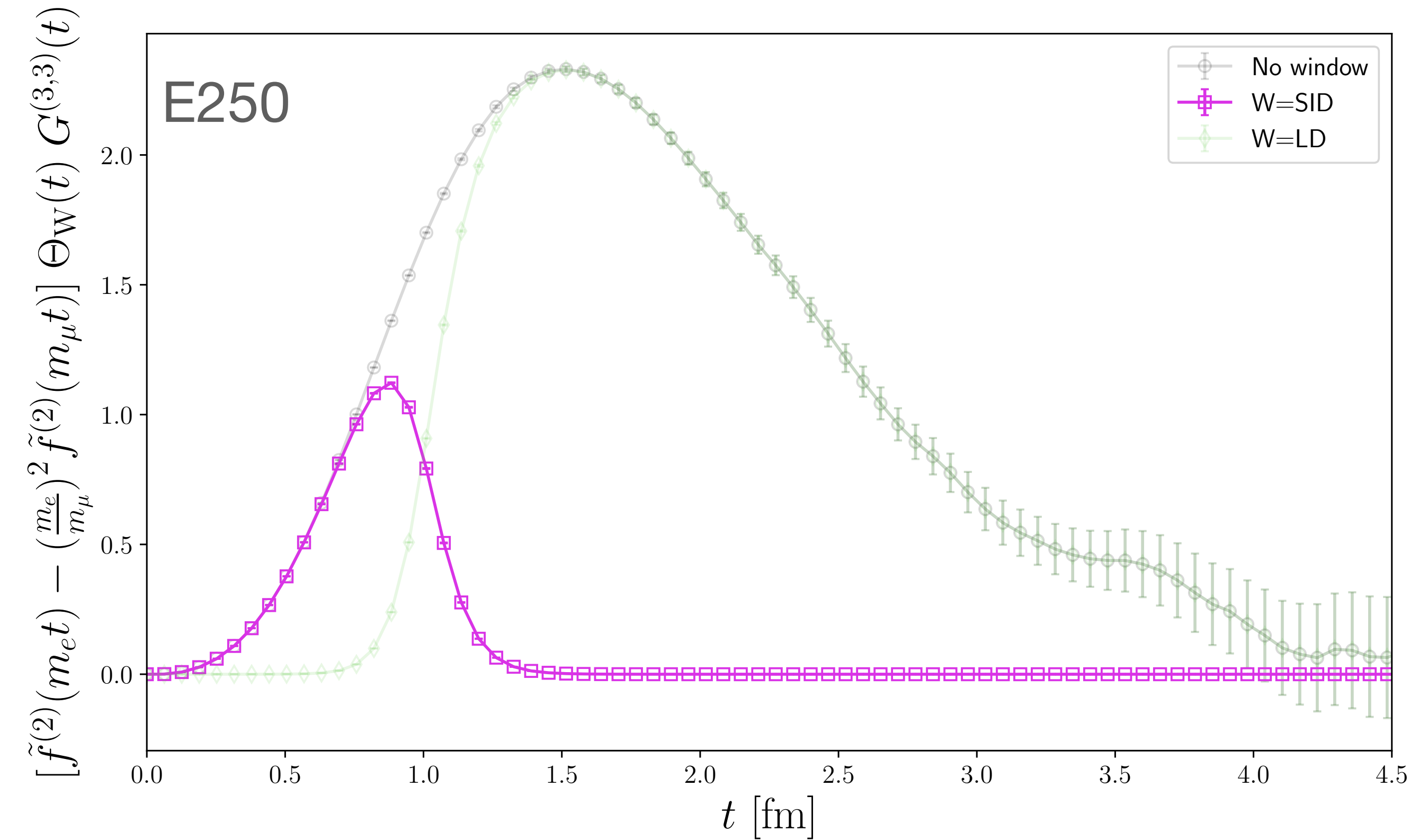
$$a_{e-\mu}^{\text{hvp}} \equiv a_e^{\text{hvp}} - \left(\frac{m_e}{m_\mu}\right)^2 a_\mu^{\text{hvp}}$$





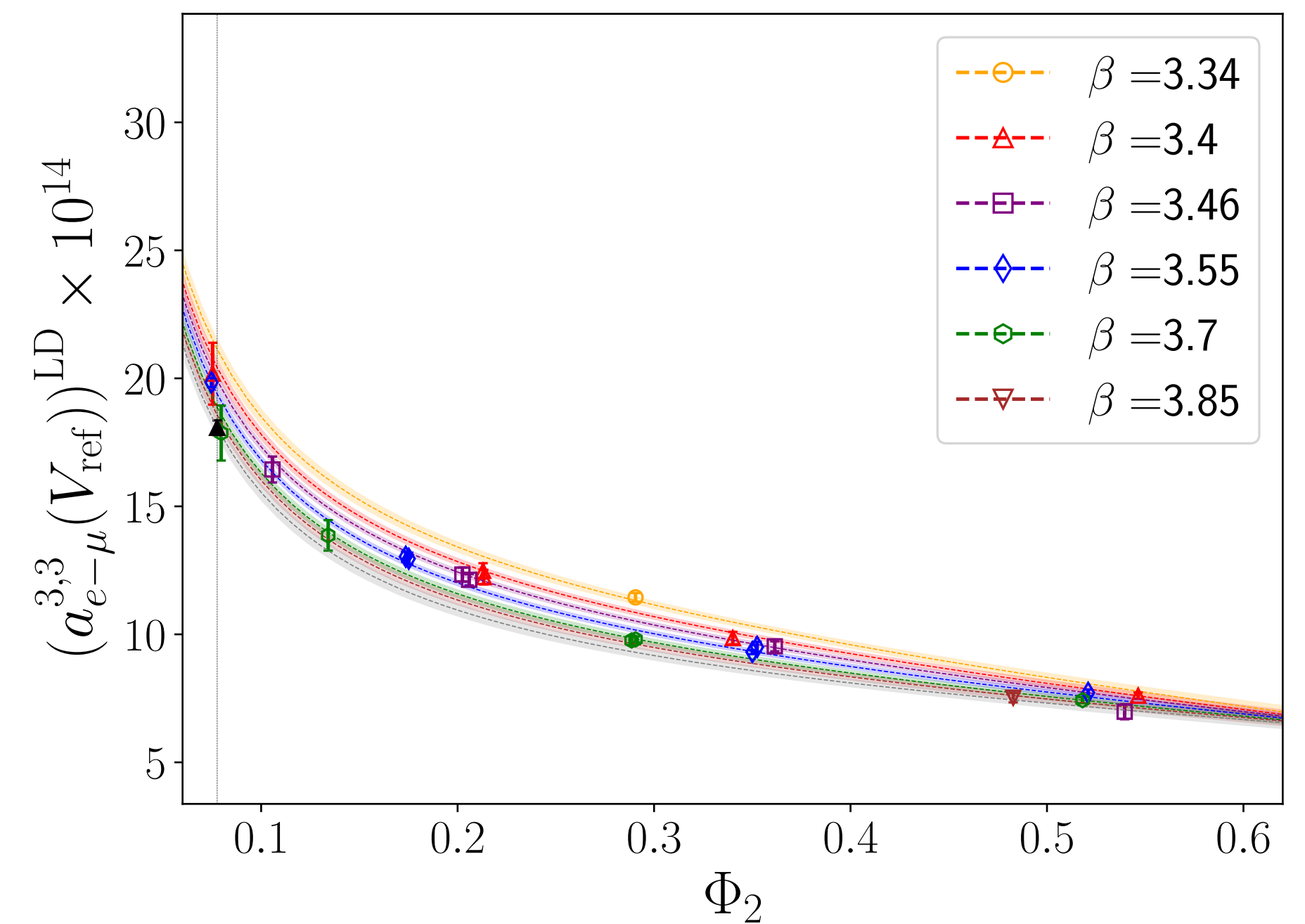
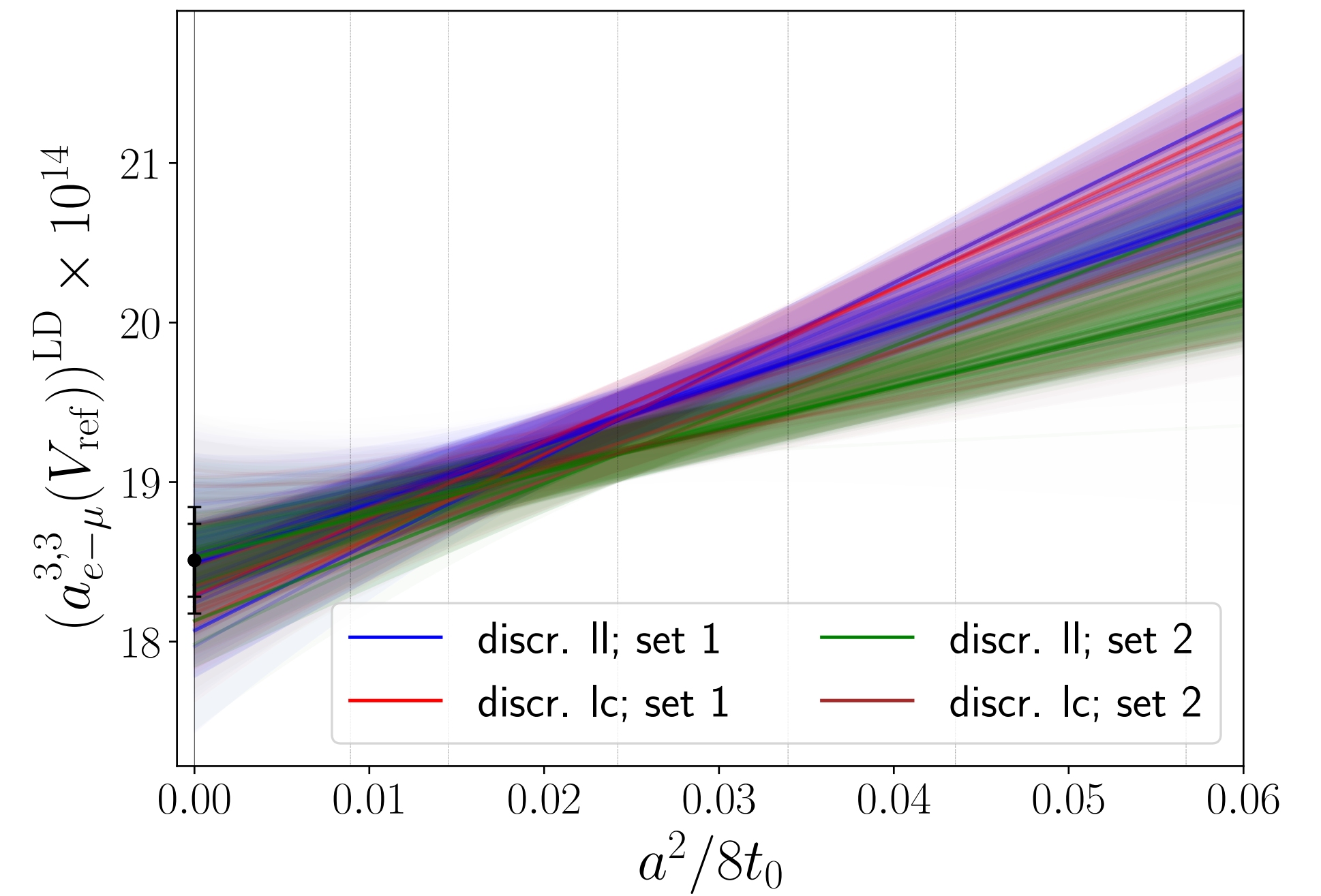
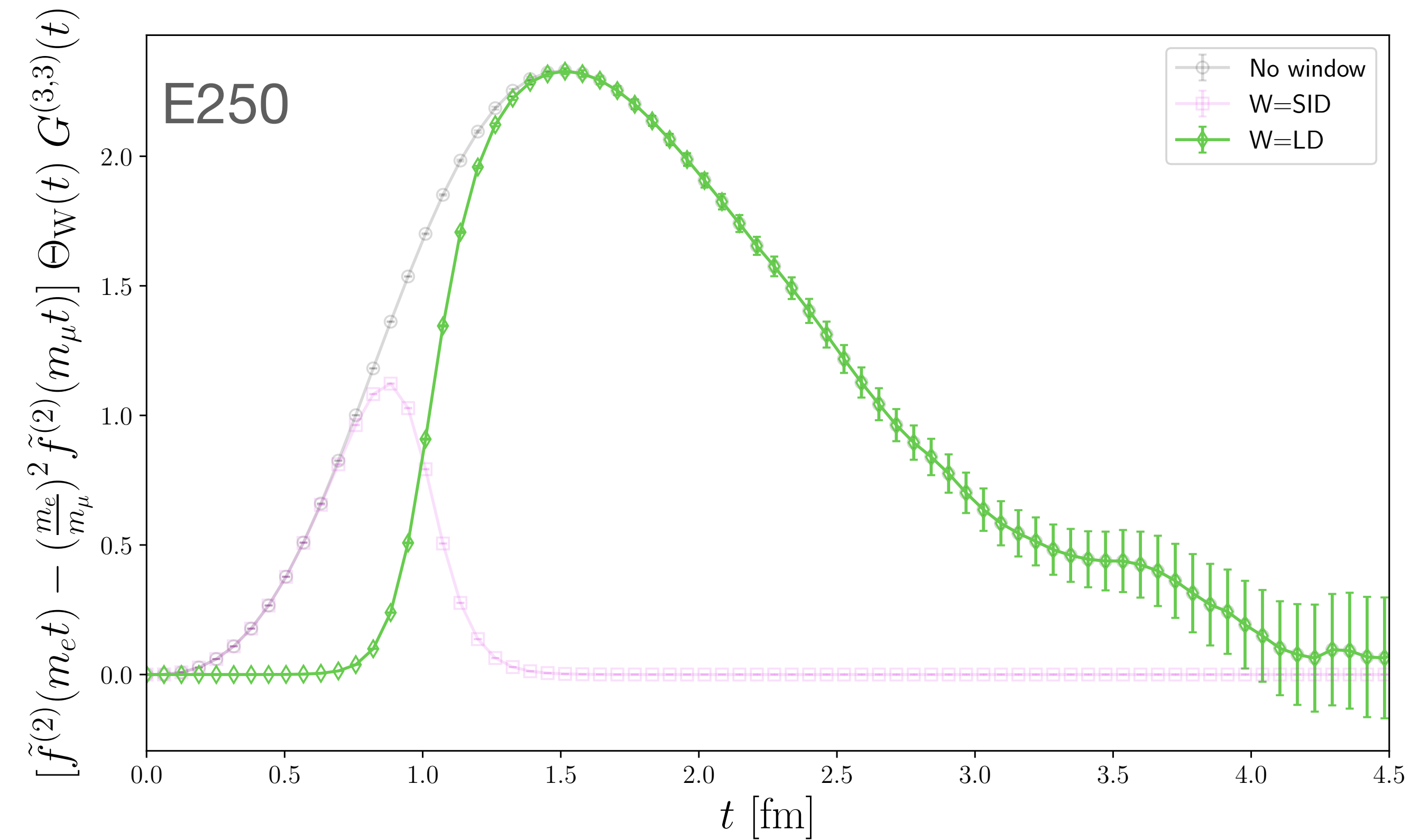
SID of $a_{e-\mu}^{\text{hvp, lo}}$

$g_e - 2$

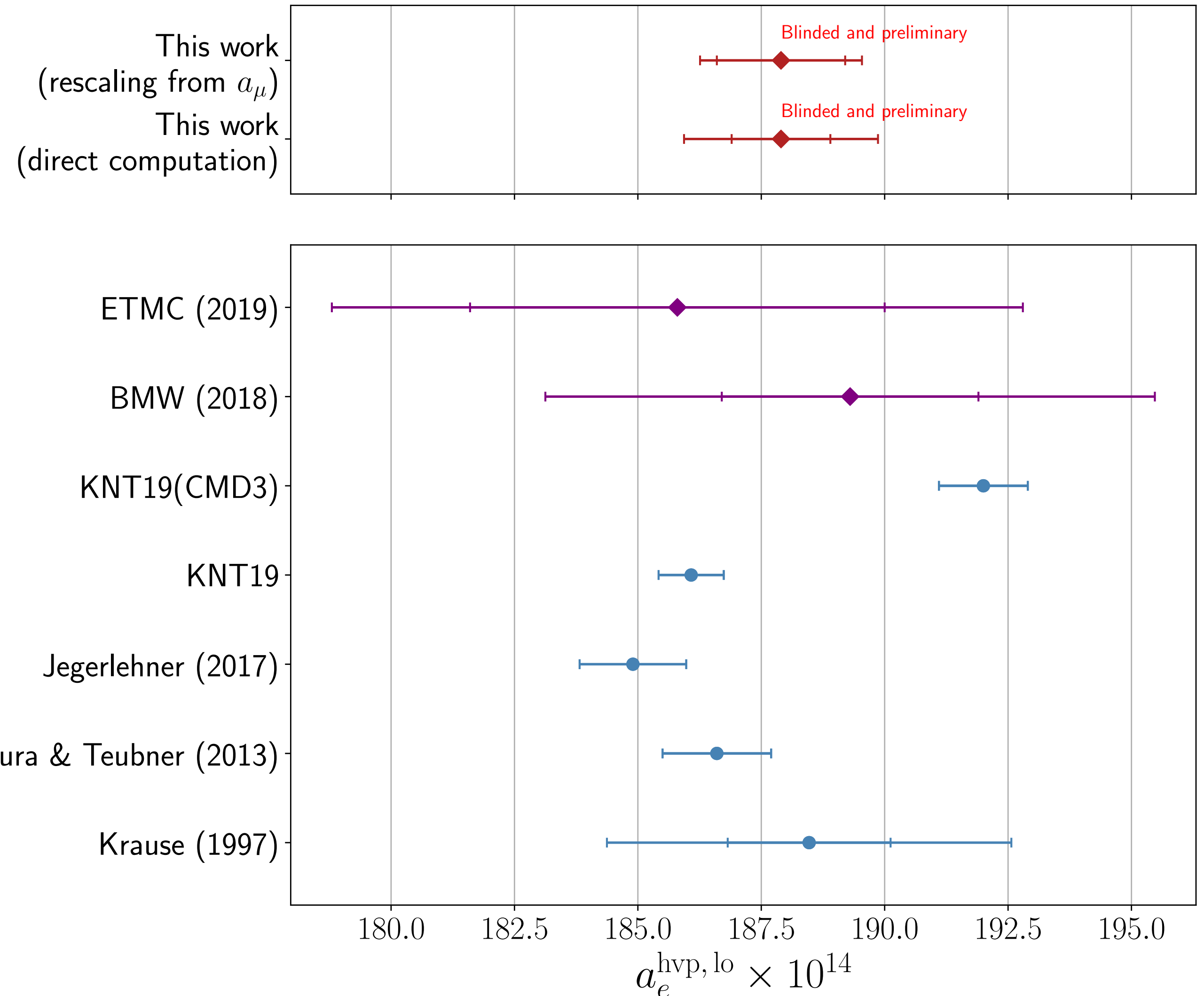


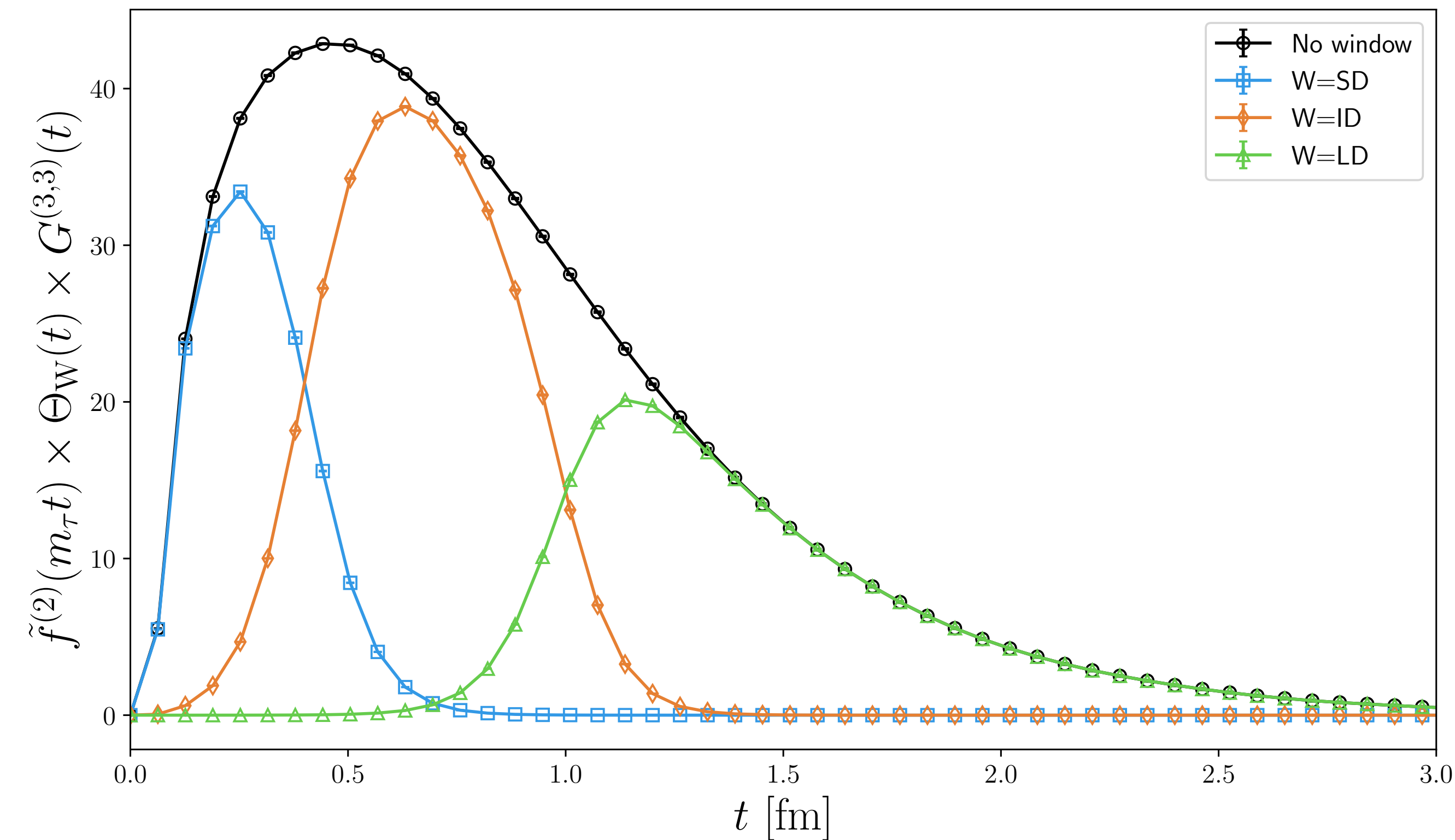
LD of $a_{e-\mu}^{\text{hvp, lo}}$

$g_e - 2$



- Calculation through $a_{e-\mu}^{\text{HVP}}$ at $\sim 1\%$ precision
- The direct computation presents larger errors - dominated by scale setting (t_0^{ph})
- IB corrections are important for a_e^{hvp} and $a_{e-\mu}^{\text{hvp}}$, still not included in this analysis
- Then, **unblinding**



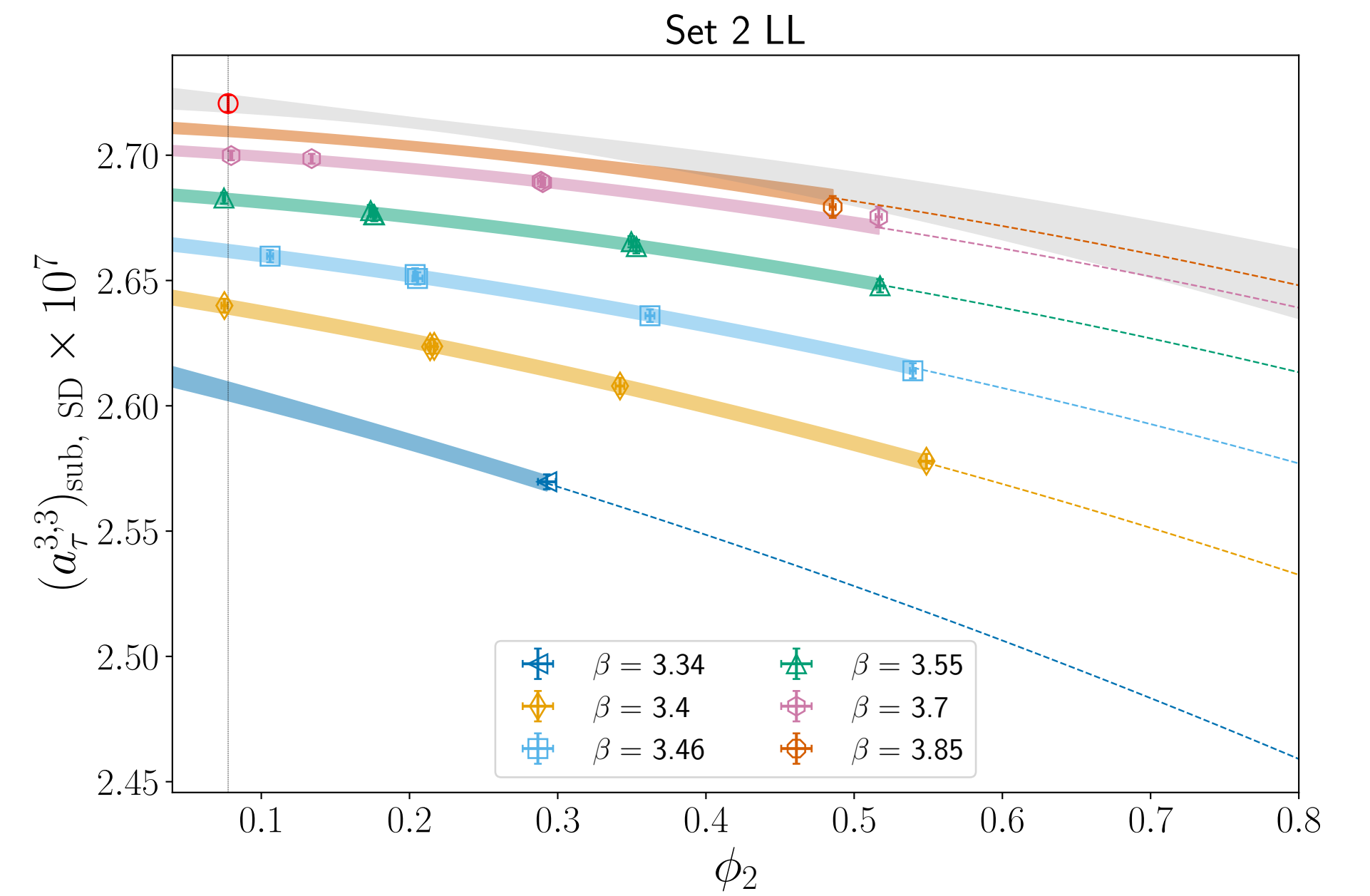
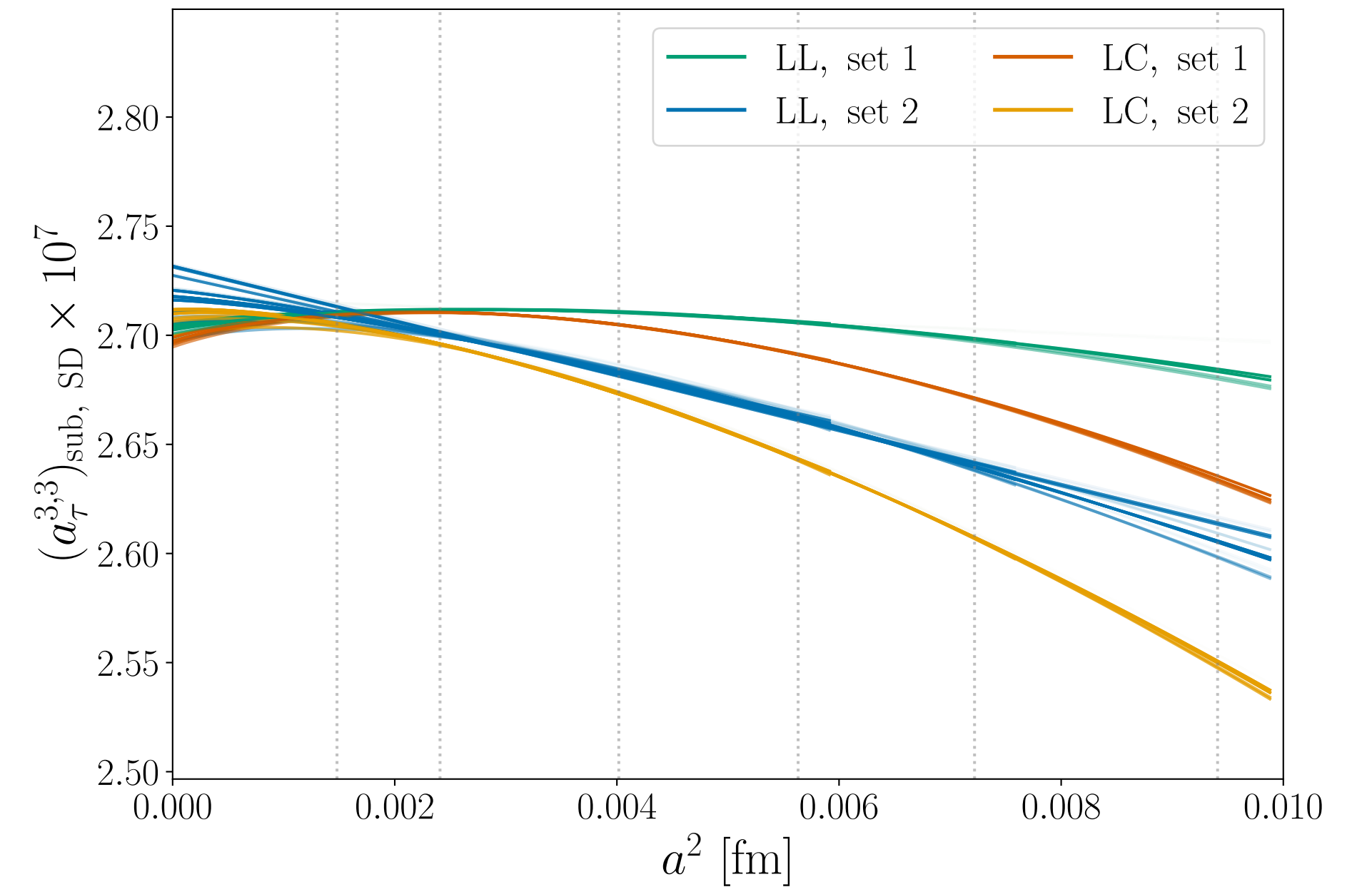
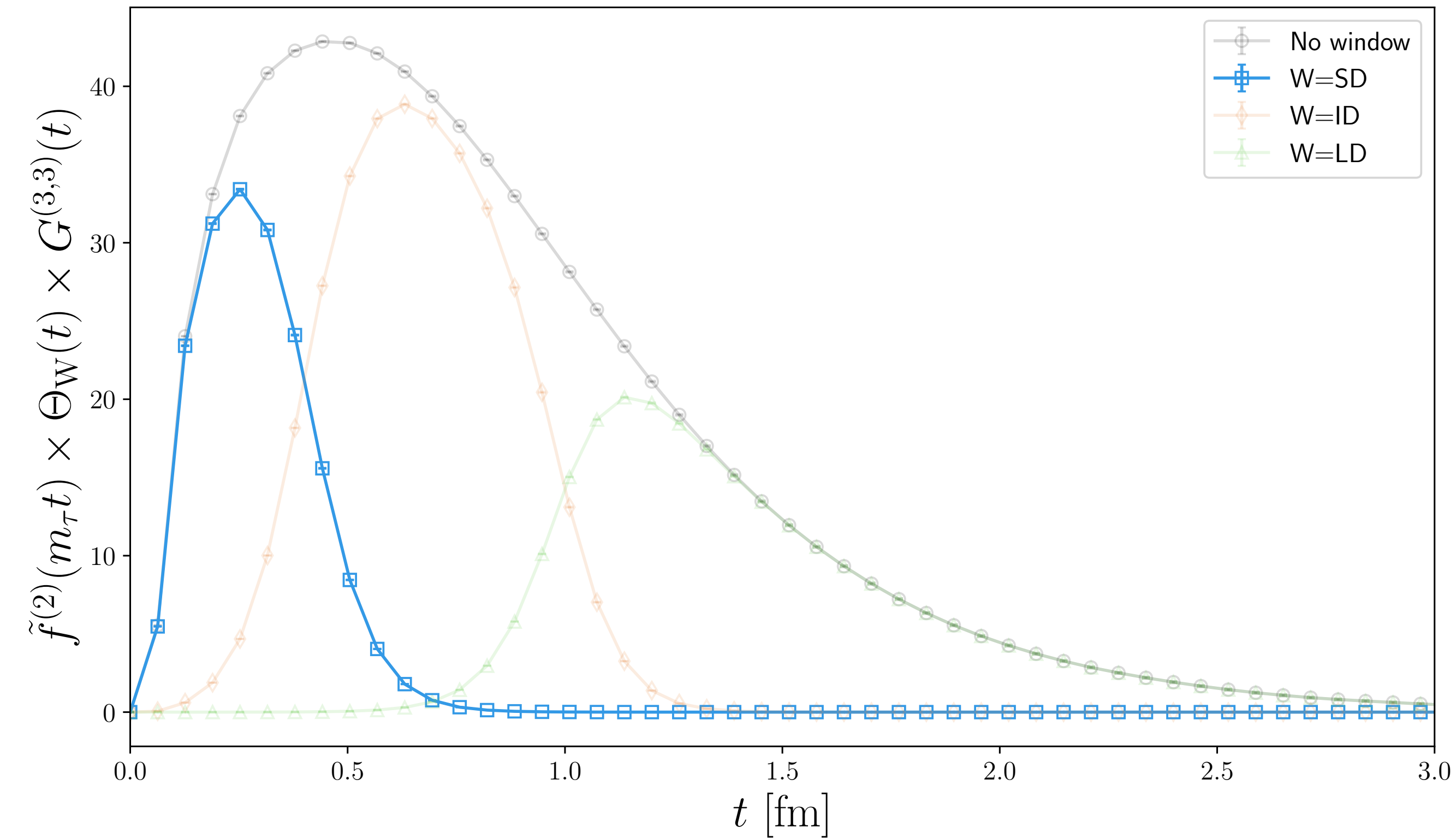


$$a_{\tau}^{\text{hvp, lo}} = \left(\frac{\alpha}{\pi} \right)^2 \int_0^{\infty} dt \tilde{f}^{(2)}(m_{\tau} t) G(t)$$

- $m_{\tau} \gg m_e, m_{\mu} \rightarrow$ High-energy observable, **more short distance**
 - Relevance of the LD window is dampened
 - Reduced systematic effects (StN, FVC, IB ...)
- Far from $\frac{m_{\ell}^2}{m_{\ell'}^2}$ behaviour
 - Rescaling from a_{μ} is not possible (*only for the SD*)
- Subtraction of the t^4 terms needed to avoid $\mathcal{O}(a^2 \ln a)$

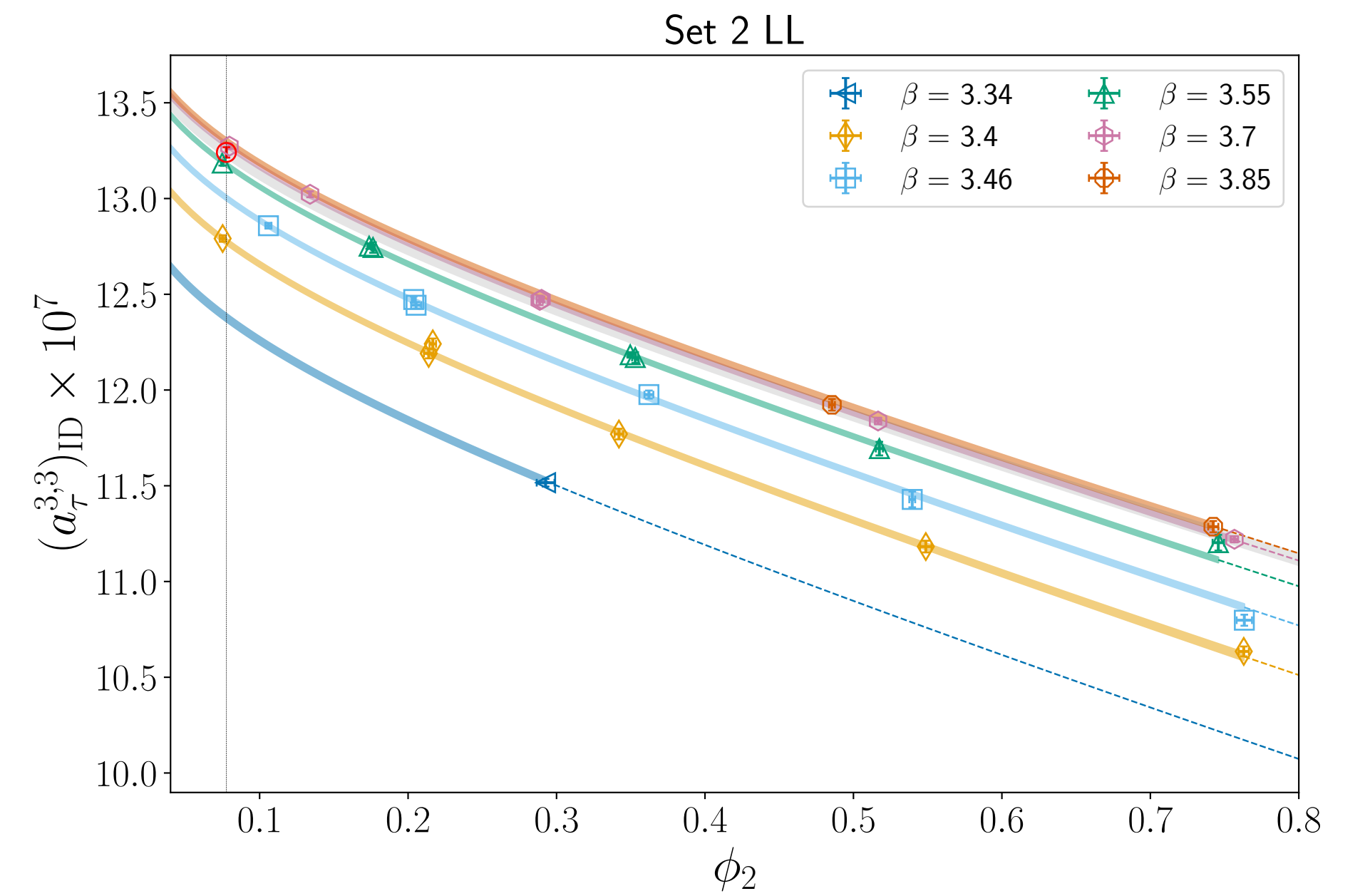
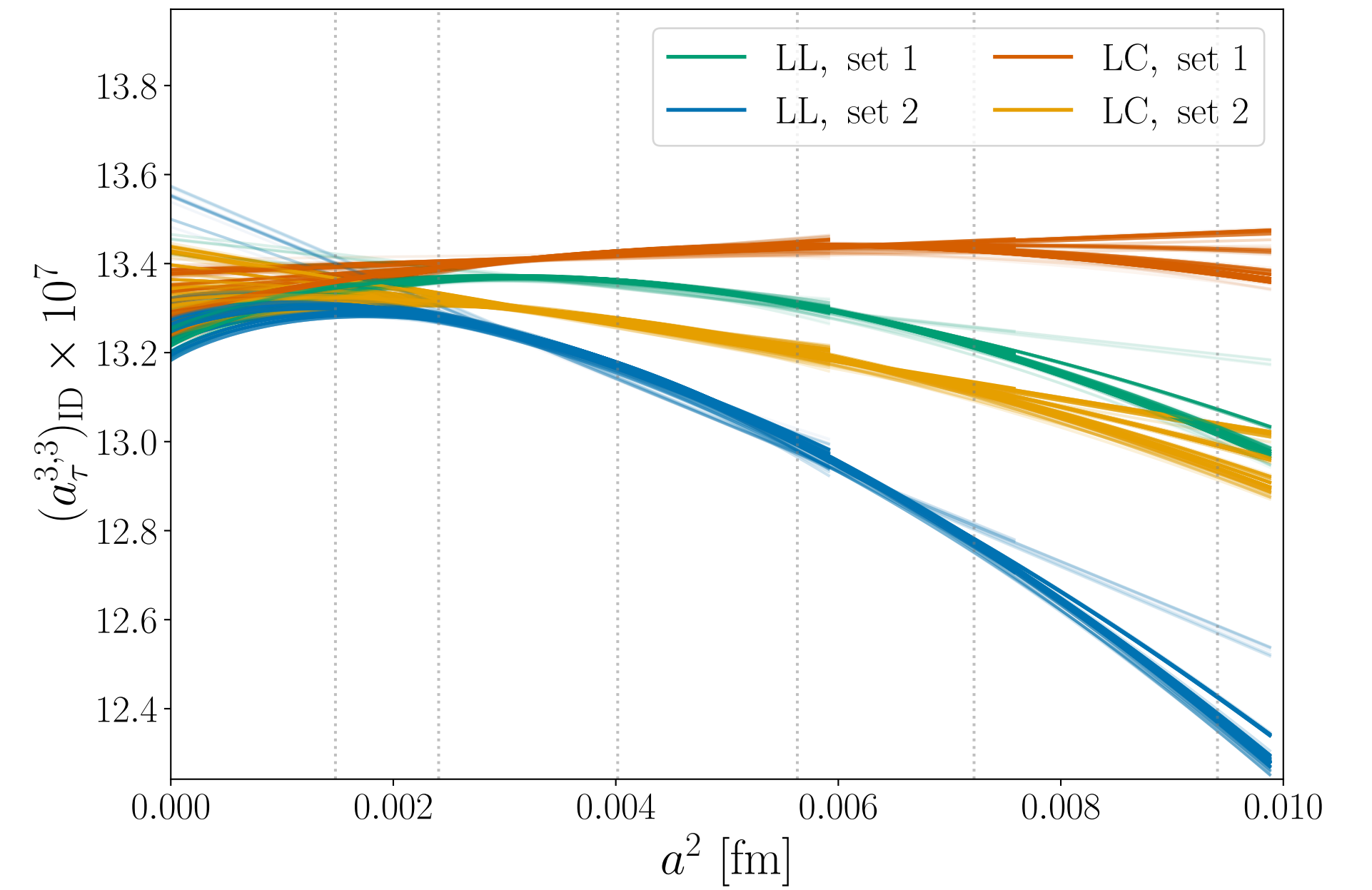
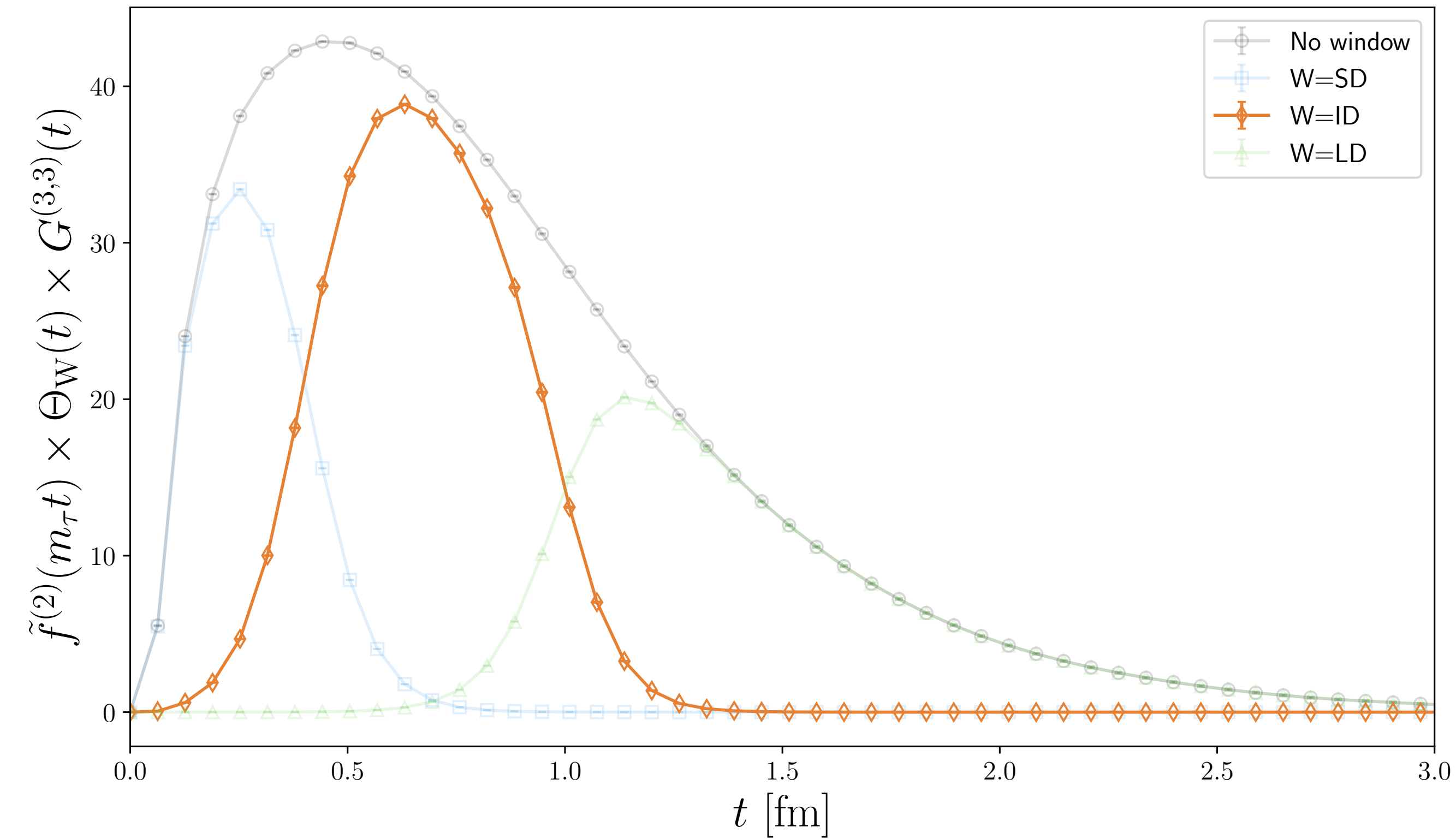
SD of $a_\tau^{\text{hvp, lo}}$

$$g_\tau - 2$$



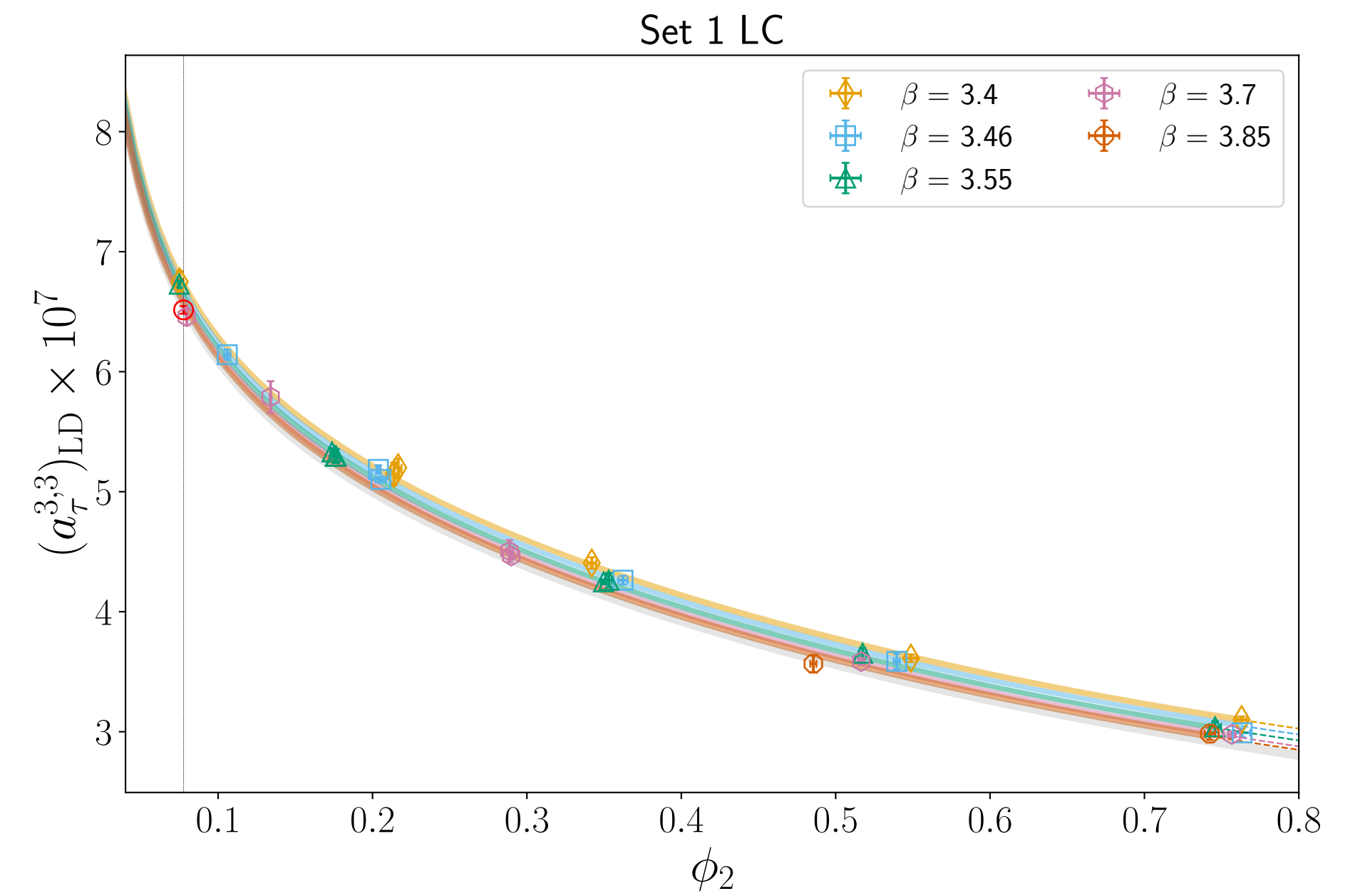
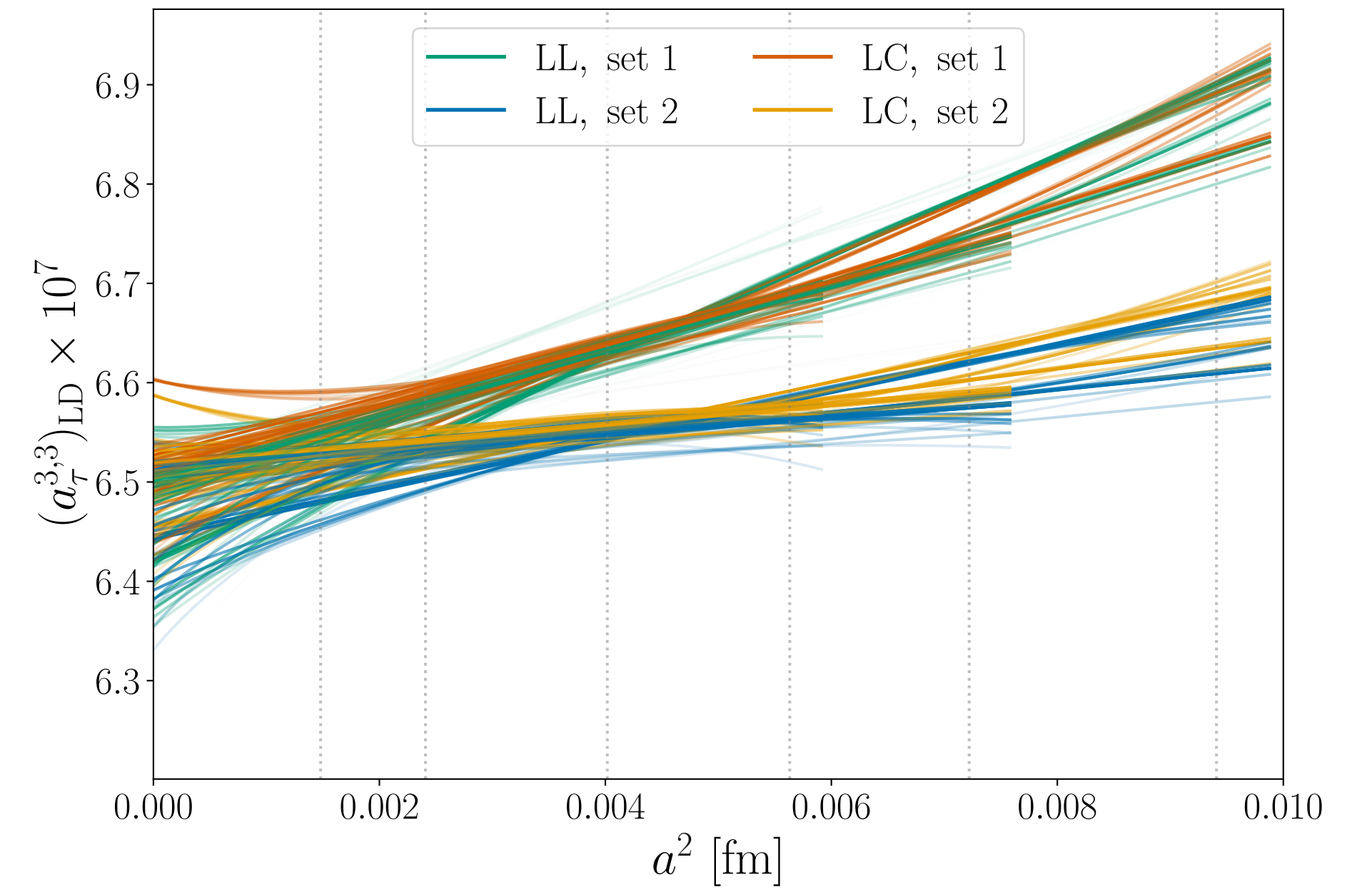
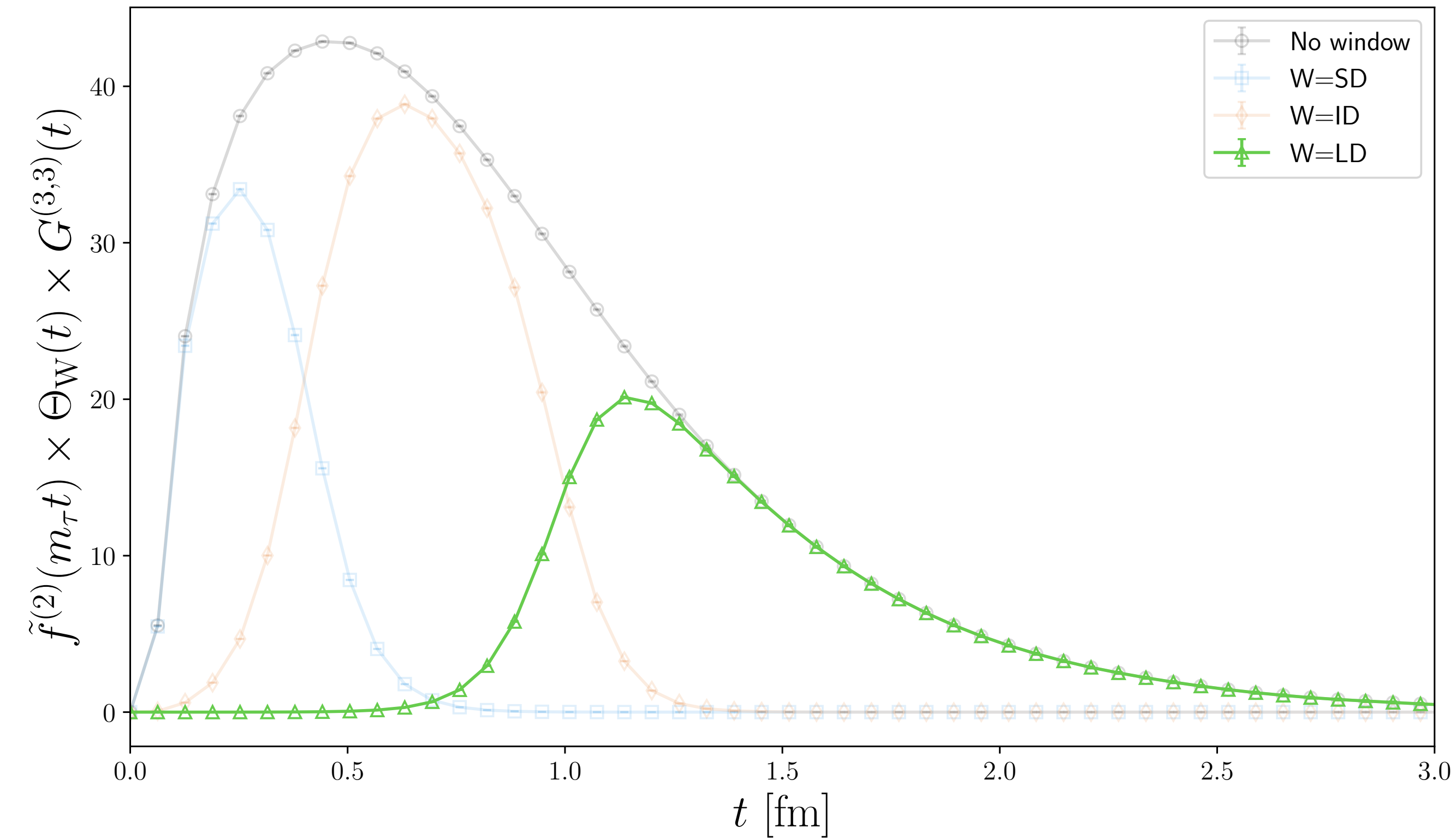
ID of $a_\tau^{\text{hvp, lo}}$

$g_\tau - 2$

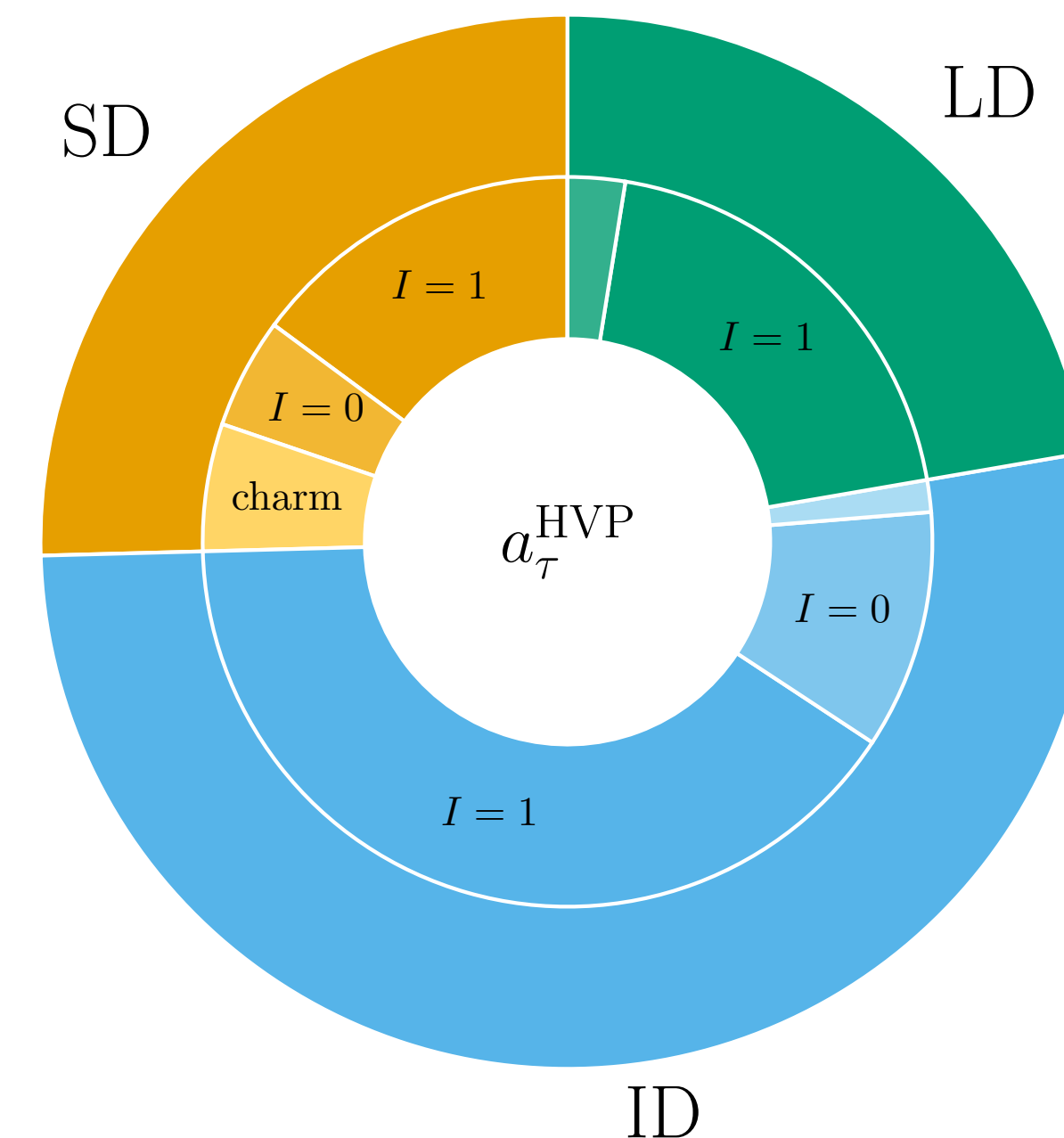
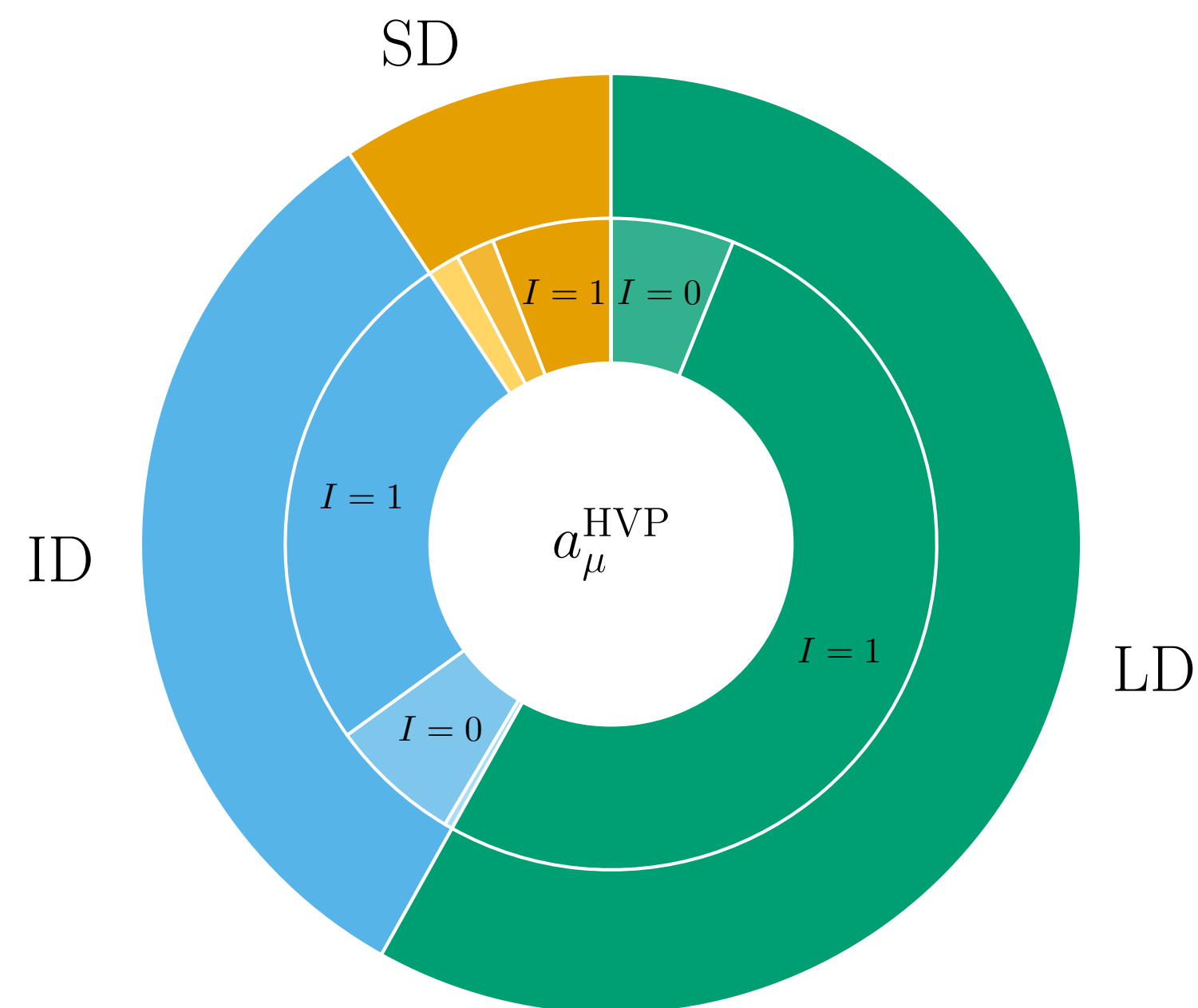


LD of $a_\tau^{\text{hvp, lo}}$

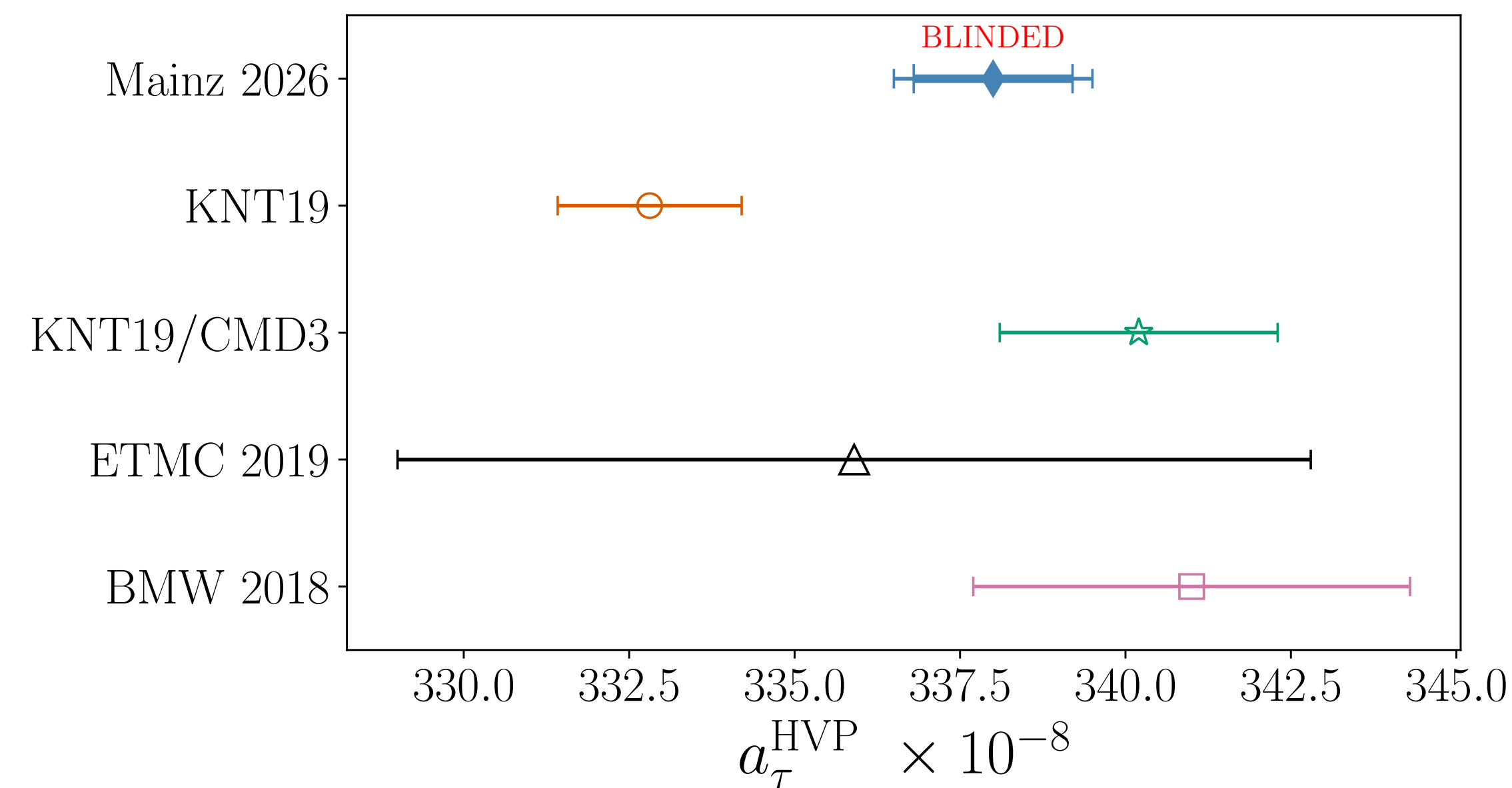
$g_\tau - 2$



- The **relevance of the LD is severely reduced** (compared to a_μ^{hvp}) $\rightarrow$ from $\sim 60\%$ to $< 25\%$
- a_τ^{hvp} is dominated by **ID window**
- The relevance of the charm channel is still subleading but its size has increased

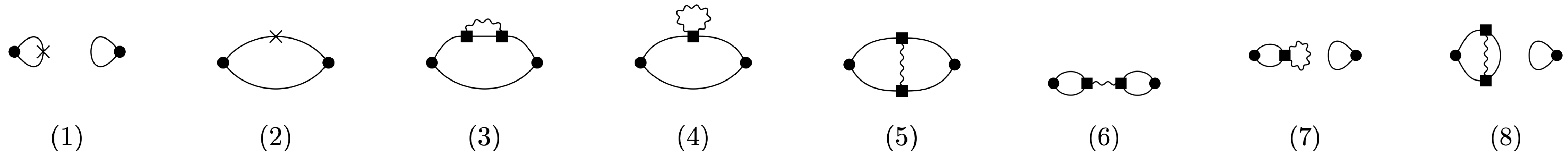


- Final result is still **blinded** and **preliminary**
- **isoQCD effects** are still to be included (*we expect them to be subleading for this observable*)
- Current error is already **competitive** with the most precise data-driven results



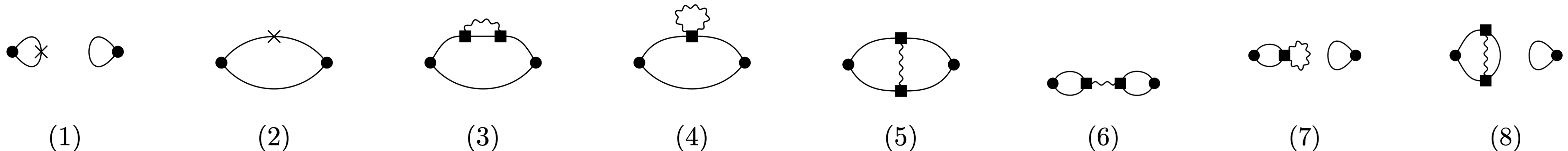
- Introduction
- IsoQCD projects
 - $\Delta\alpha^{\text{had}}(Q^2)$
 - Higher-order HVP in a_μ
 - Other lepton flavors
- Isospin breaking corrections

- As our flagship approach, we proceed with the [RM123](#) method. We have currently two programs following this approach
 - [QED_L](#) making use of the [TMR](#)
 - Computation through the [CCS method](#), using a [Pauli-Villars regulated](#) photon



- Regarding the strong IB corrections, we also explore its effects dynamically
 - We have recently generated a $1 + 1 + 1$ ensemble ([Y450](#))

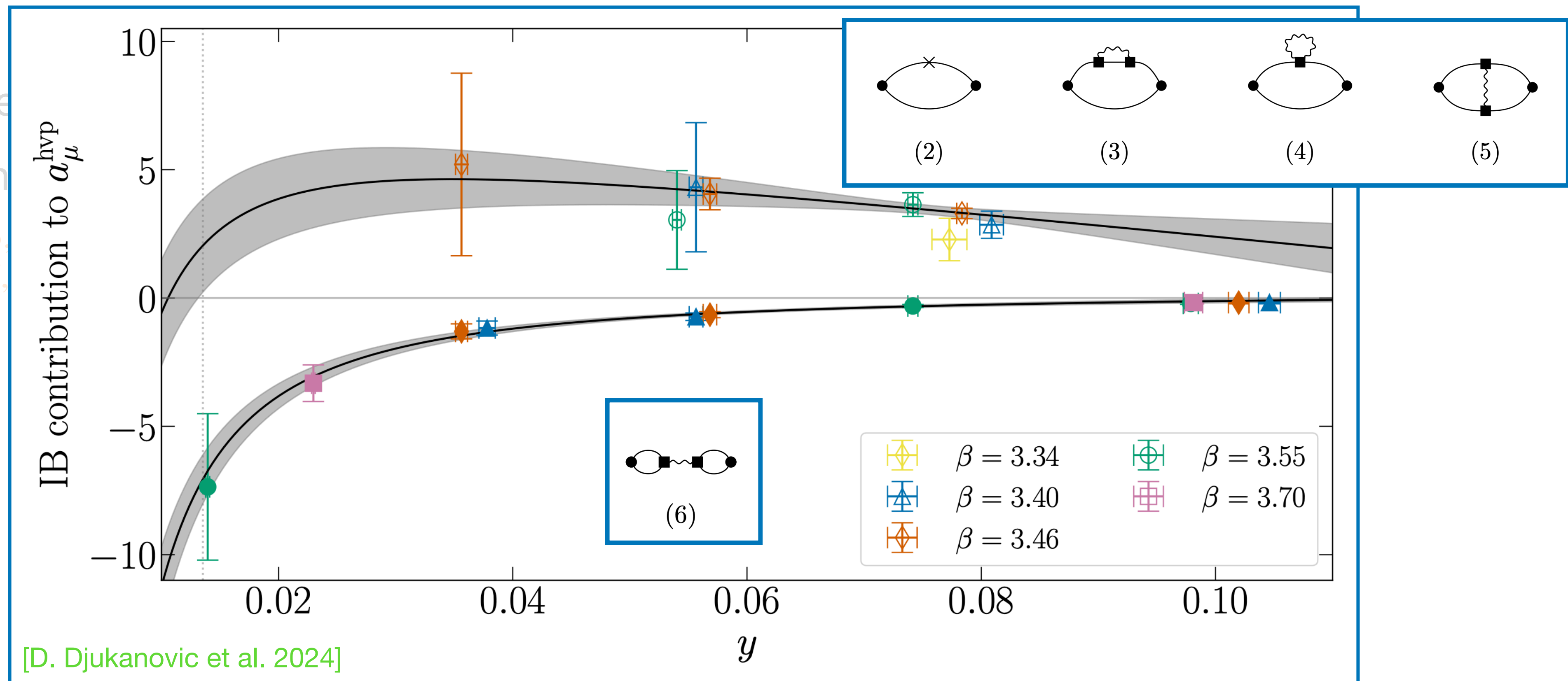
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$$a_{\mu}^{\text{HVP}} = a_{\mu}^{3,3} + \frac{1}{3}a_{\mu}^{8,8} + \frac{2}{\sqrt{3}}a_{\mu}^{3,8} + \dots$$

- We place spe
- $G^{3,8} = 0$ in
- $a_{\ell}^{3,8} = a_{\ell}^{3,8}$



$$a_{\mu}^{\text{HVP}} = a_{\mu}^{3,3} + \frac{1}{3}a_{\mu}^{8,8} + \frac{2}{\sqrt{3}}a_{\mu}^{3,8} + \dots$$

- We place special focus on the **3,8 channel** - *isospin violating piece*

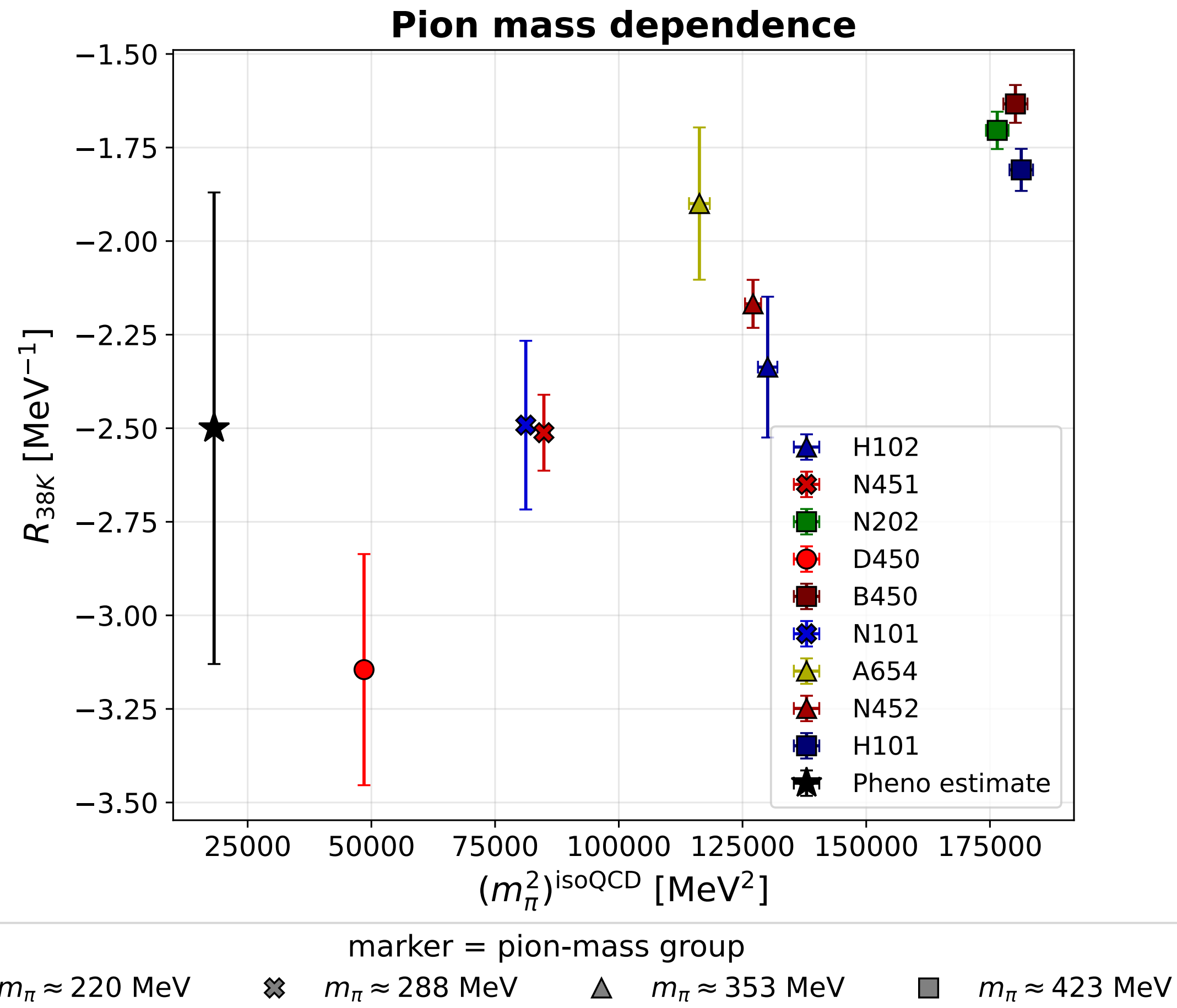
- $G^{3,8} = 0$ in the isoQCD limit

- $a_{\ell}^{3,8} = a_{\ell,\text{em}}^{3,8} + \Delta^{\delta m} M_K R_{38K} = a_{\ell,\text{em}}^{3,8} + (\Delta^{\text{ph}} M_K - \Delta^{\text{em}} M_K) R_{38K}$

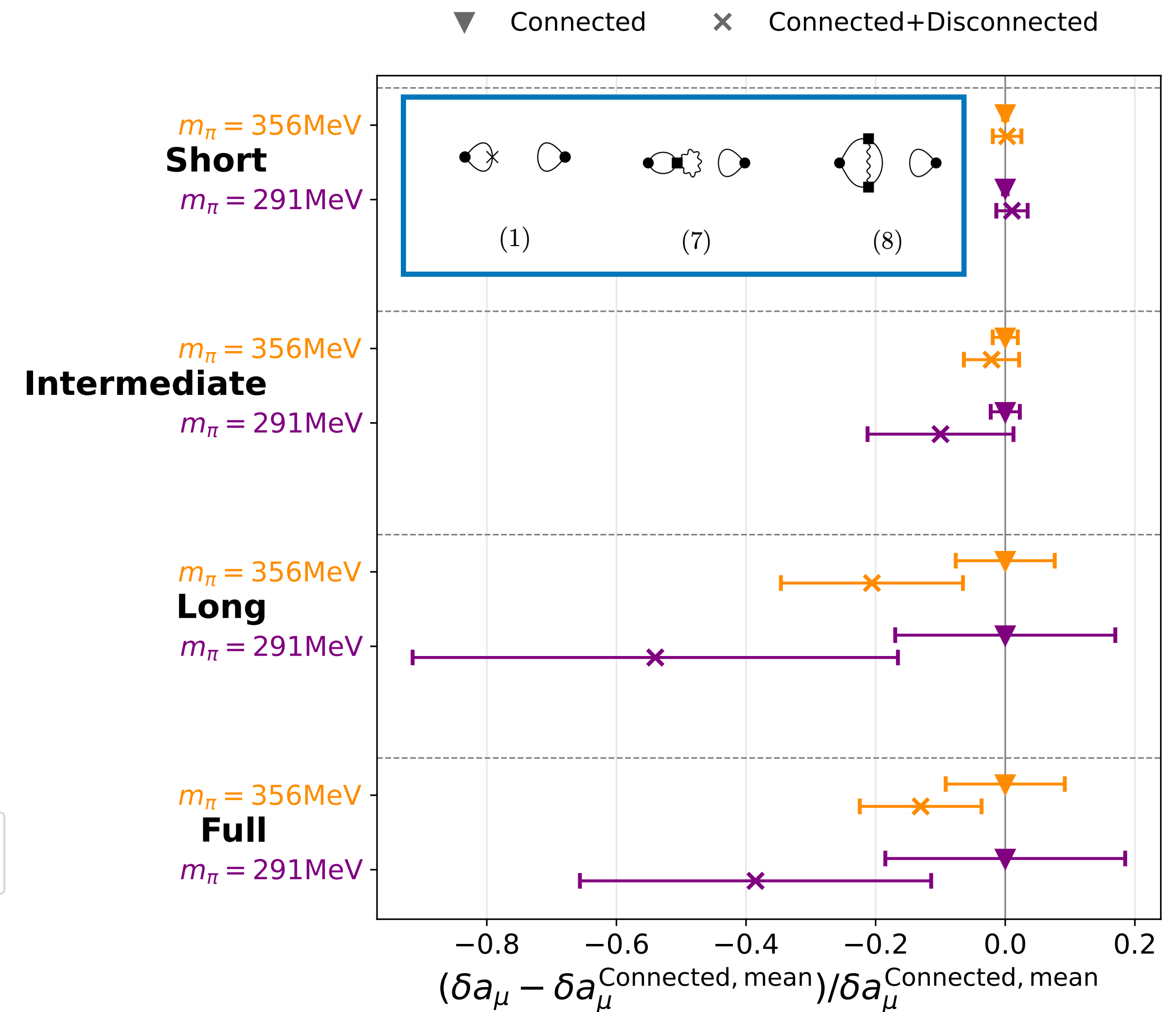
Electromagnetic

Strong isospin breaking

$$R_{38K} = \frac{\partial a_{\mu}^{3,8}}{\partial(m_u - m_d)} \left(\frac{\partial M_K}{\partial(m_u - m_d)} \right)^{-1} \text{ is scheme independent}$$



See S. Lahrtz talk: Wednesday 16:30 - 16:45



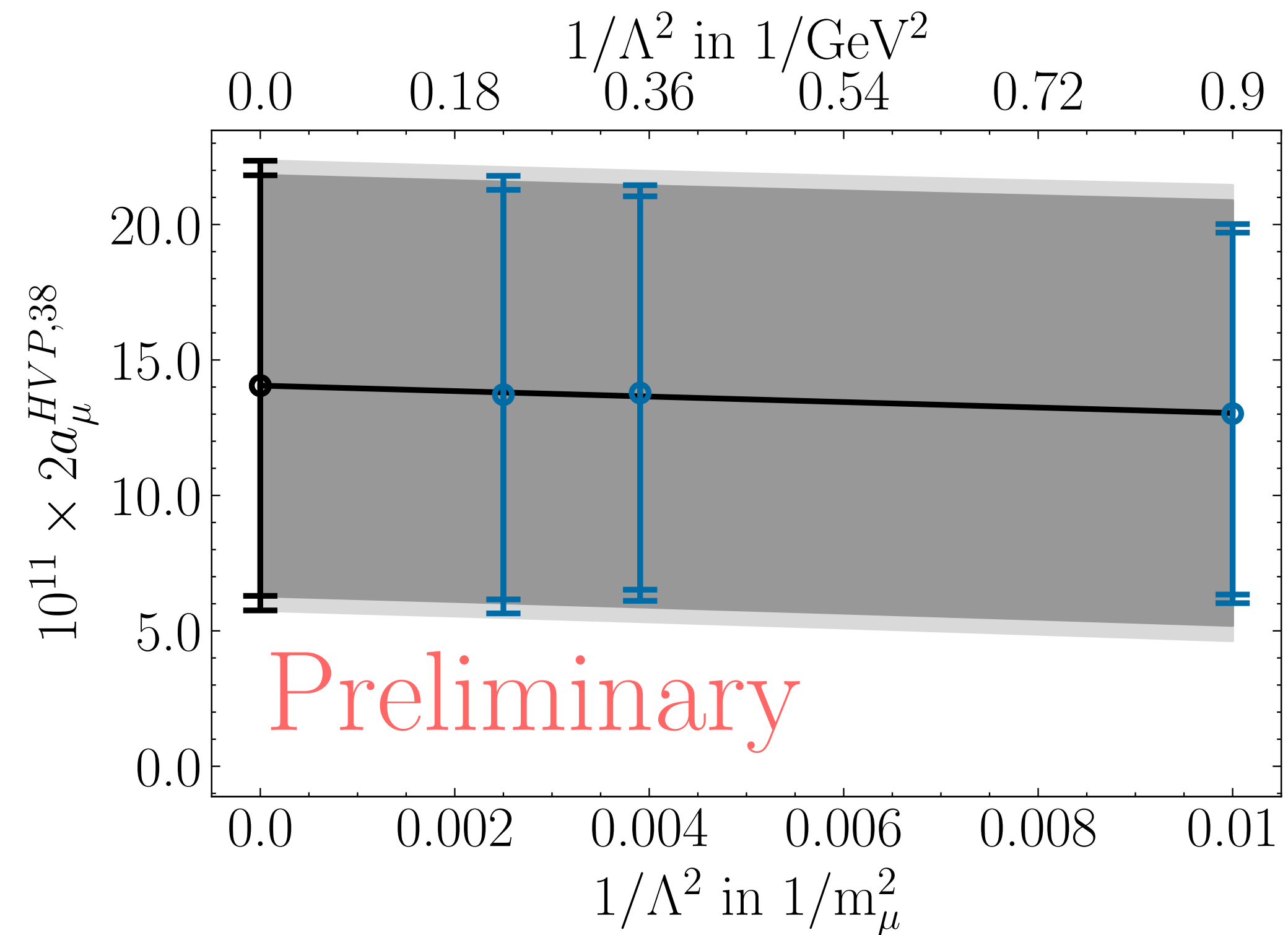
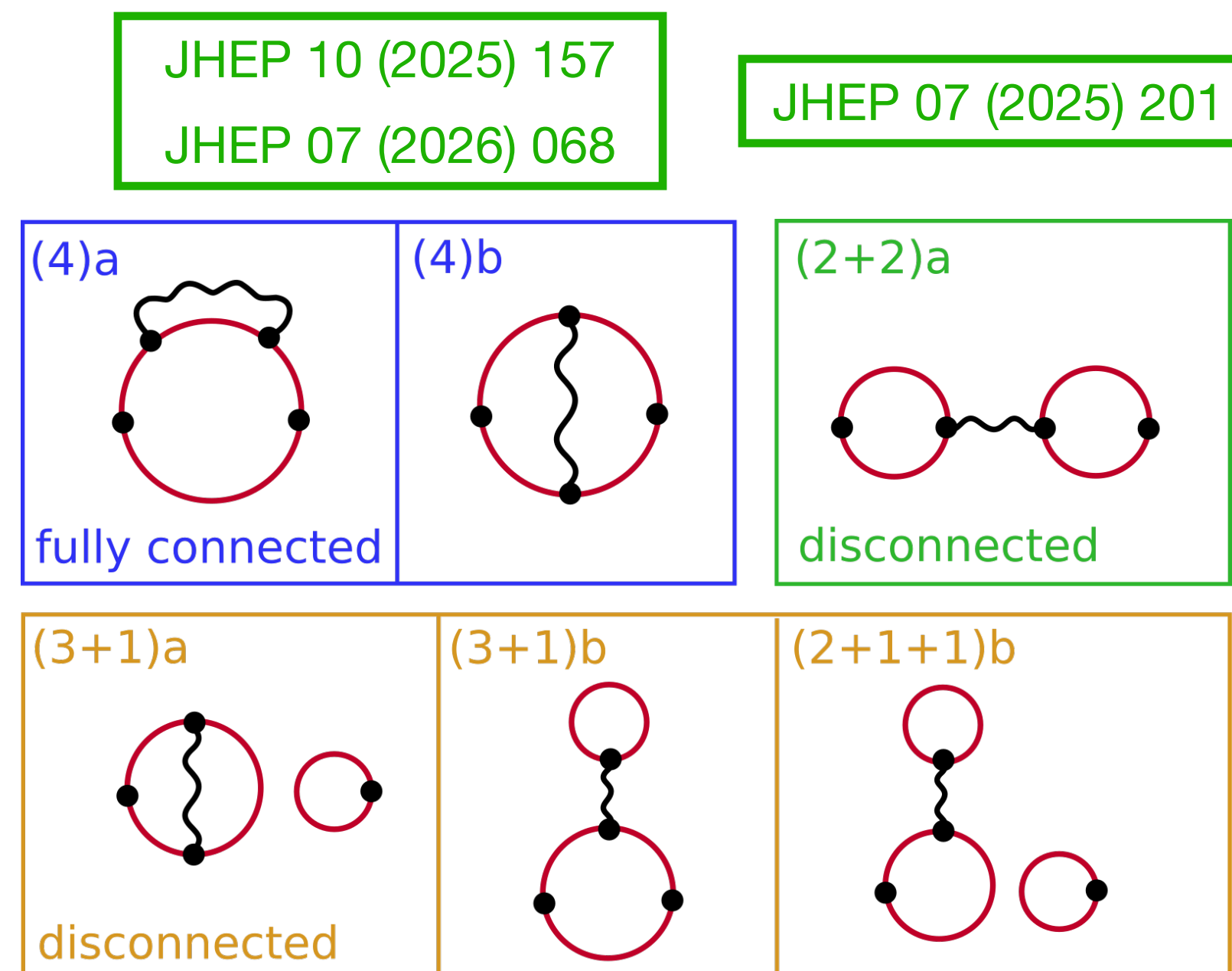
In the Covariant-Coordinate Space (CCS) method the **electromagnetic contribution** to $a_\mu^{3,8}$ is written as

$$a_{\mu,\text{em}}^{3,8} = -\frac{e^2}{2} \int_{x,y,z} H_{\lambda\sigma}(z) [\mathcal{G}(x-y)]_\Lambda \langle j_\lambda^3(z) j_\mu^{\text{em}} j_\nu^{\text{em}} j_\sigma^8 \rangle + C_T(\Lambda)$$

With a PV-regulated photon propagator

$$[\mathcal{G}(x)]_\Lambda = \frac{1}{4\pi^2 |x|^2} + G_\Lambda^{(1)}(x) - 2G_{\frac{\Lambda}{\sqrt{2}}}^{(1)}(x) \quad \text{where} \quad G_m^{(1)}(x) = \frac{mK_1(m|x|)}{4\pi^2 |x|}$$

and $\Lambda \ll 1/a$



Conclusions

- We have total **analytical control on the time-kernels at NLO** for all Euclidean times
 - Manuscript in preparation
 - Working package already developed “hvpKernels.jl”
- Estimate for the **NLO HVP to the muon $g - 2$** is now complete and accepted for publication
- Progress on a complete determination of the **LO HVP to the electron and tau $g - 2$**
 - a_e^{HVP} : Rescaling from a_μ working at $\sim 1\%$ relative precision. Still room for improvement: **scale-setting dominates the error**
 - a_τ^{HVP} : Direct isoQCD computation yields $\lesssim 0.5\%$ precision
- **Parallel isospin breaking projects**
 - $\text{QED}_L + \text{TMR}$
 - $\gamma^{\text{PV}} + \text{CCS}$
 - Crosscheck on RM123 through a $1 + 1 + 1$ ensemble

Related HVP Mainz talks:

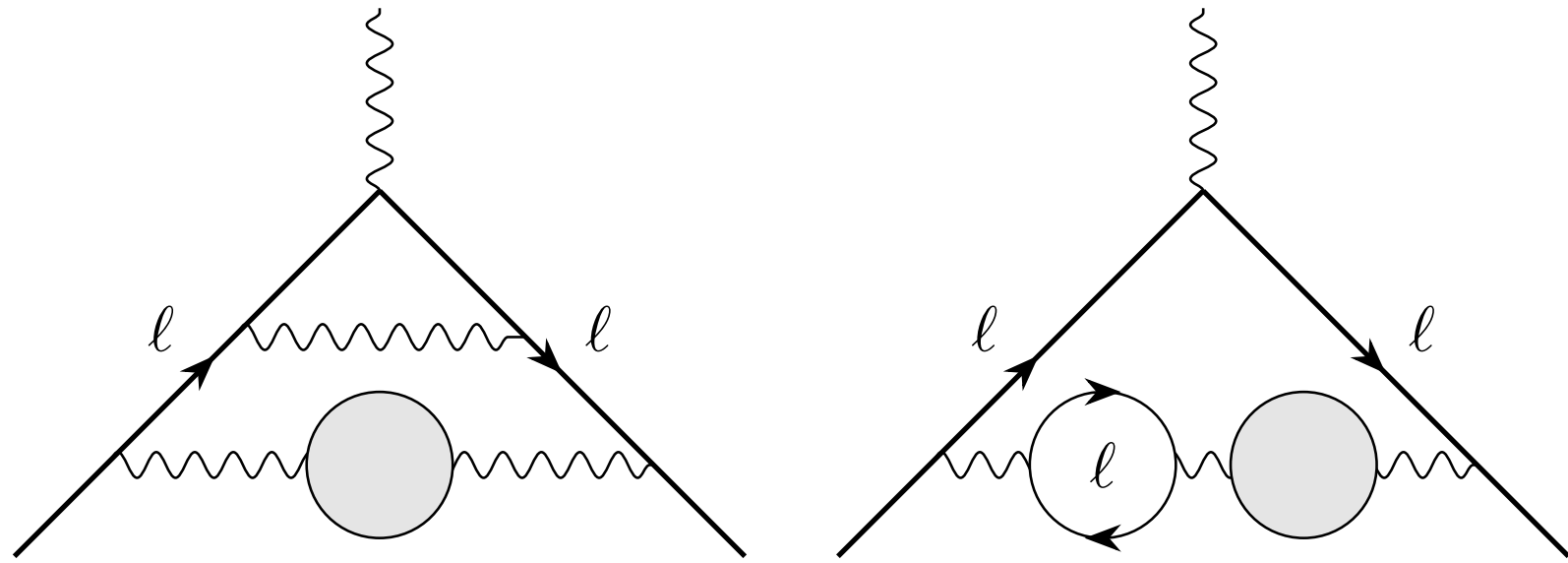
- S. Lahrtz : Wed. 16:30 - 16:45
- A. Conigli : Thu. 10:15 - 10:45

Other Mainz talks:

- D. Erb : Thu. 11:45 - 12:15

Backup slides

- IsoQCD projects
 - Higher-order HVP in a_μ
 - Other lepton flavors
- Isospin breaking corrections



$$\frac{m_\ell^2}{8\pi^2} \tilde{f}^{(4a)}(\hat{t}) = \int_0^\infty \frac{d\hat{\omega}}{\hat{\omega}} \hat{f}^{(4a)}(\hat{\omega}^2) \left[\hat{\omega}^2 \hat{t}^2 - 4 \sin^2 \frac{\hat{\omega} \hat{t}}{2} \right]$$

[S. Laporta et al. 2024]

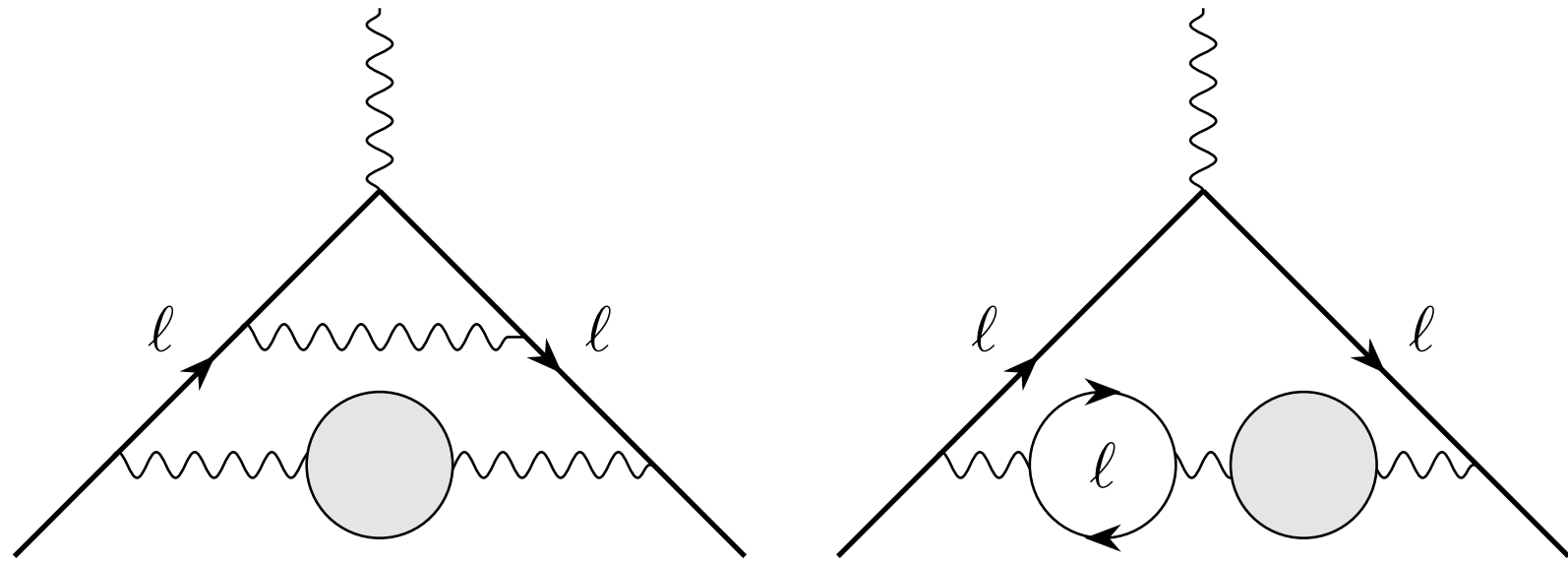
$$\frac{m_\ell^2}{16\pi^2} \tilde{f}^{(4a)}(\hat{t}) \stackrel{\hat{t} \ll 1}{\equiv} \sum_{\substack{n \geq 4 \\ n \text{ even}}} \frac{\hat{t}^n}{n!} \left(a_n + b_n \pi^2 + c_n (\gamma_E + \ln \hat{t}) + d_n (\gamma_E + \ln \hat{t})^2 \right)$$

Dominant piece: $\mathcal{D} \sim \hat{t}^2$

Suppressed piece: $\mathcal{S}$

*Can be used to interpolate between
 $\hat{t} \ll 1$ and $\hat{t} \gg 1$*

$$\frac{m_\ell^2}{16\pi^2} \tilde{f}^{(4a)}(\hat{t}) \stackrel{\hat{t} \gg 1}{\equiv} \mathcal{D}(\hat{t}) + \mathcal{S}(\hat{t})$$



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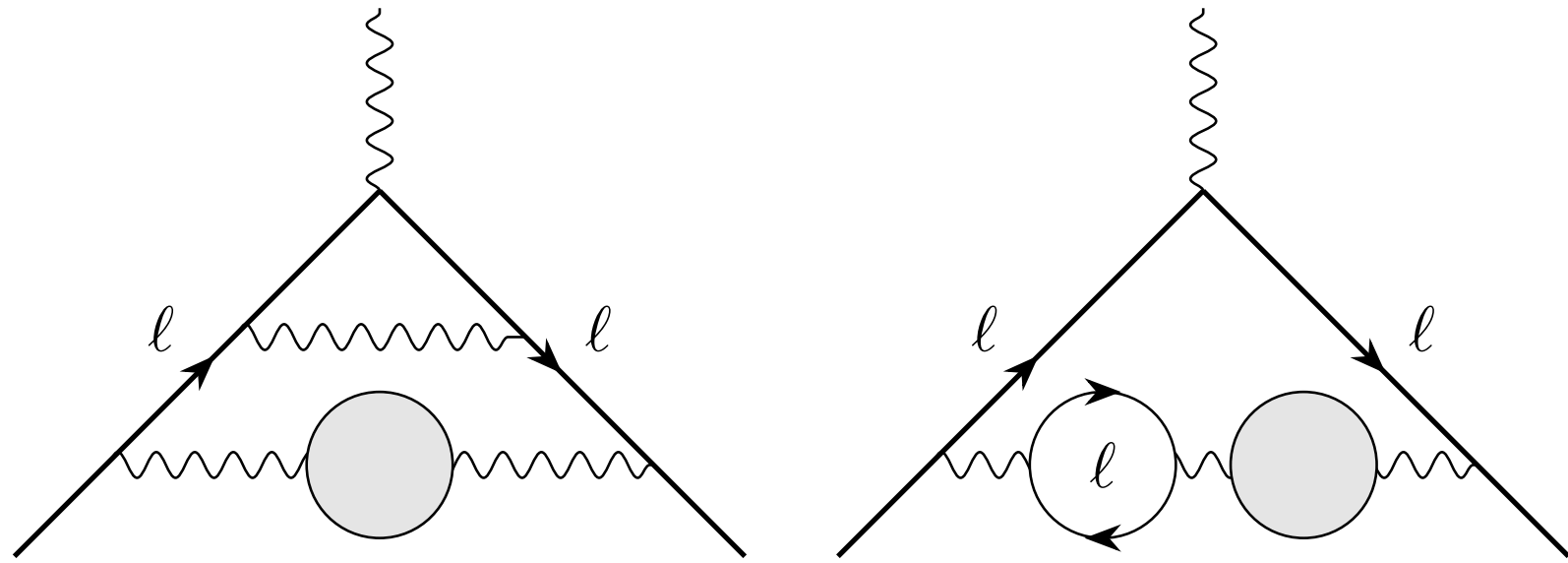
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[S. Laporta et al. 2024]

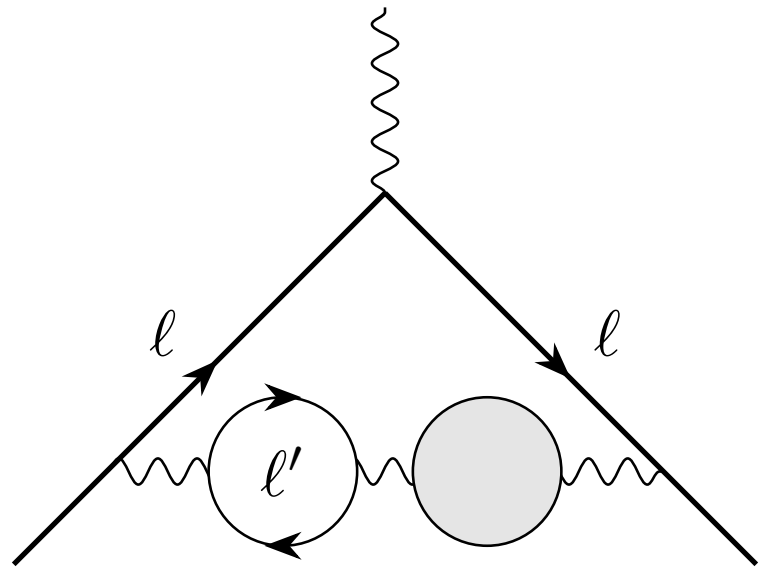
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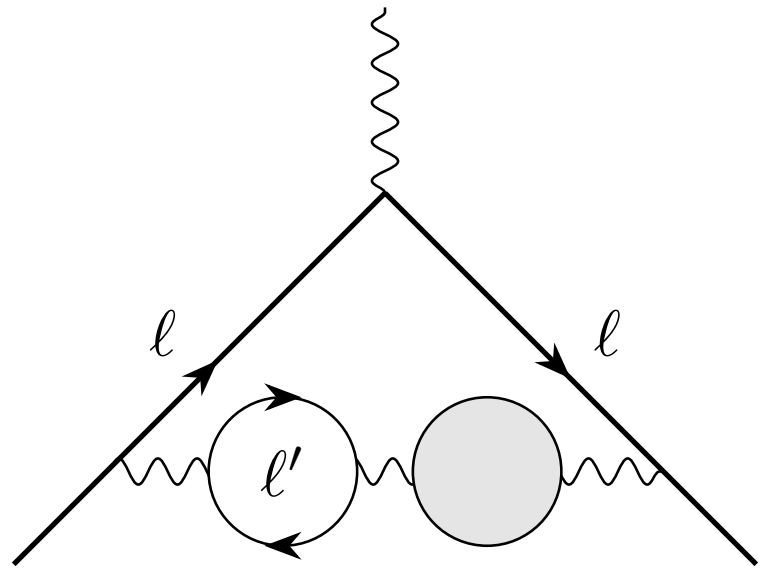
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$$M \equiv \frac{m_{\ell'}}{m_\ell} \ll 1$$

$$\frac{m_\ell^2}{16\pi^2} \tilde{f}_{\ell'}^{(4b)}(\hat{t}) \stackrel{\hat{t} \ll 1}{\equiv} \sum_{n \geq 4} \sum_{m \geq 0} \frac{\hat{t}^n}{n!} \left(\frac{m_{\ell'}}{m_\ell} \right)^m \left(e_{nm} + f_{nm} \pi^2 + g_{nm} (\gamma_E + \ln \hat{t}) + h_{nm} (\gamma_E + \ln \hat{t})^2 \right)$$

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THREE REGIONS: $\hat{t} \ll 1 \ll 1/M$, $1 \ll \hat{t} \ll 1/M$ AND $1 \ll 1/M \ll \hat{t}$



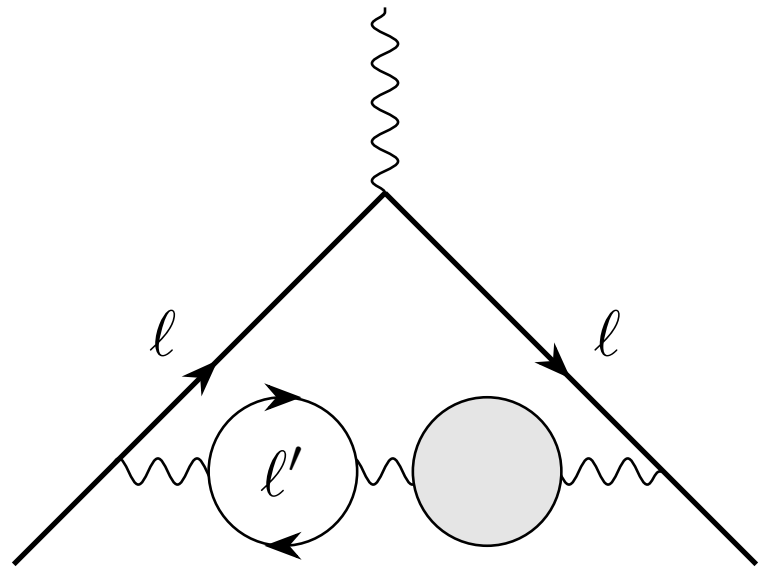
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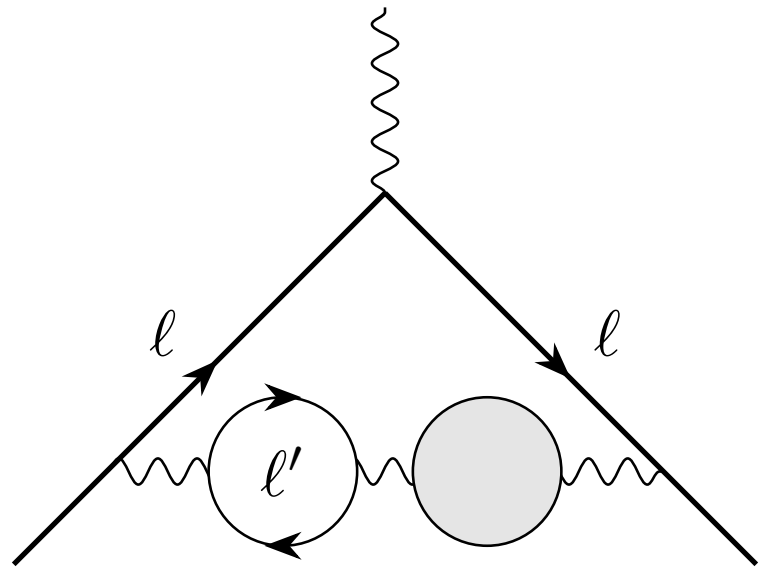
$$\frac{m_\ell^2}{8\pi^2} \tilde{f}^{(4b)}(\hat{t}) = 2 \int_0^\infty \frac{d\hat{\omega}}{\hat{\omega}} \hat{f}^{(2)}(\hat{\omega}^2) F_l(\omega; m_{\ell'}) \left[\hat{\omega}^2 \hat{t}^2 - 4 \sin^2 \frac{\hat{\omega} \hat{t}}{2} \right]$$

$$M \equiv \frac{m_{\ell'}}{m_\ell} \ll 1$$

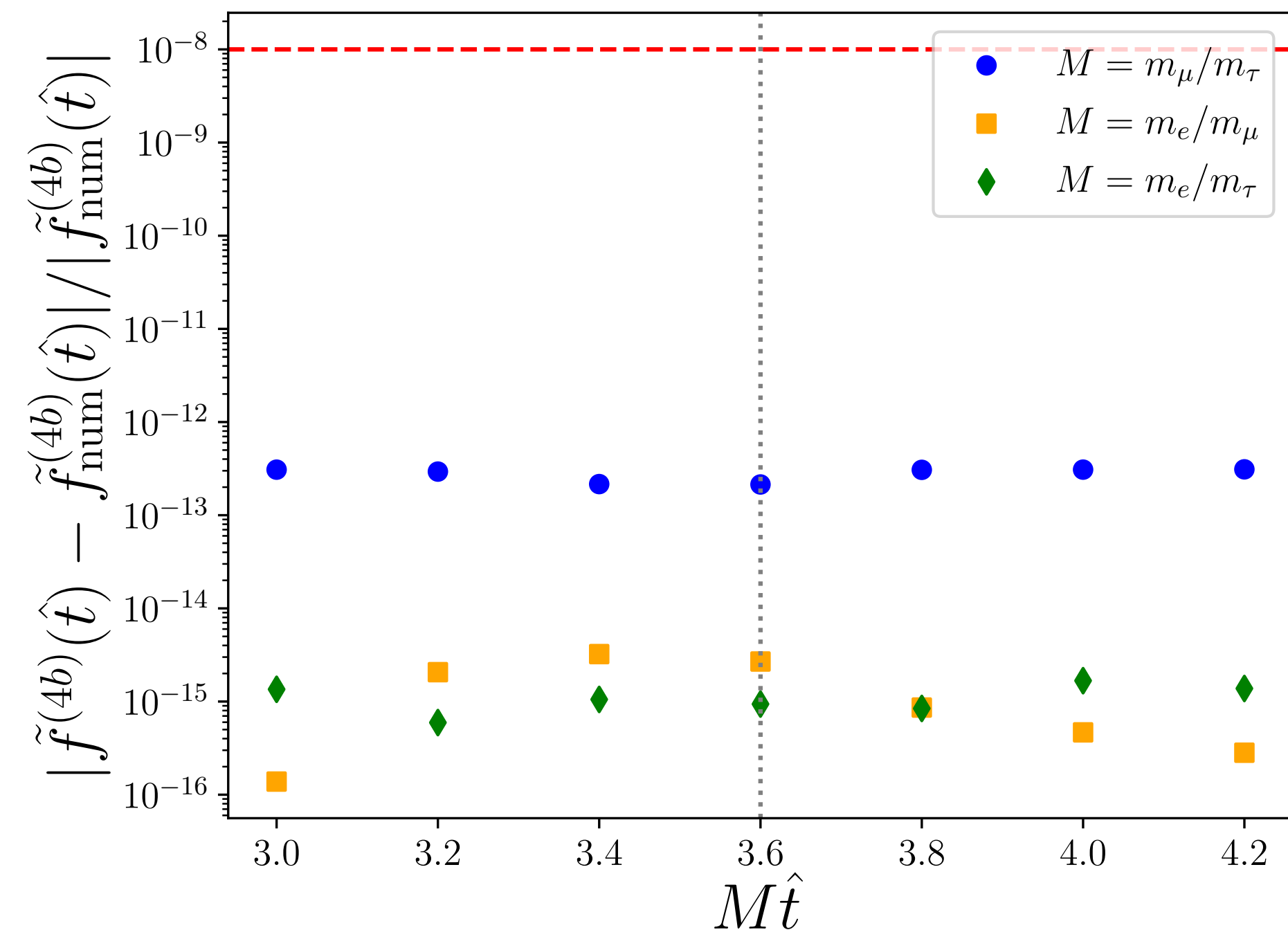
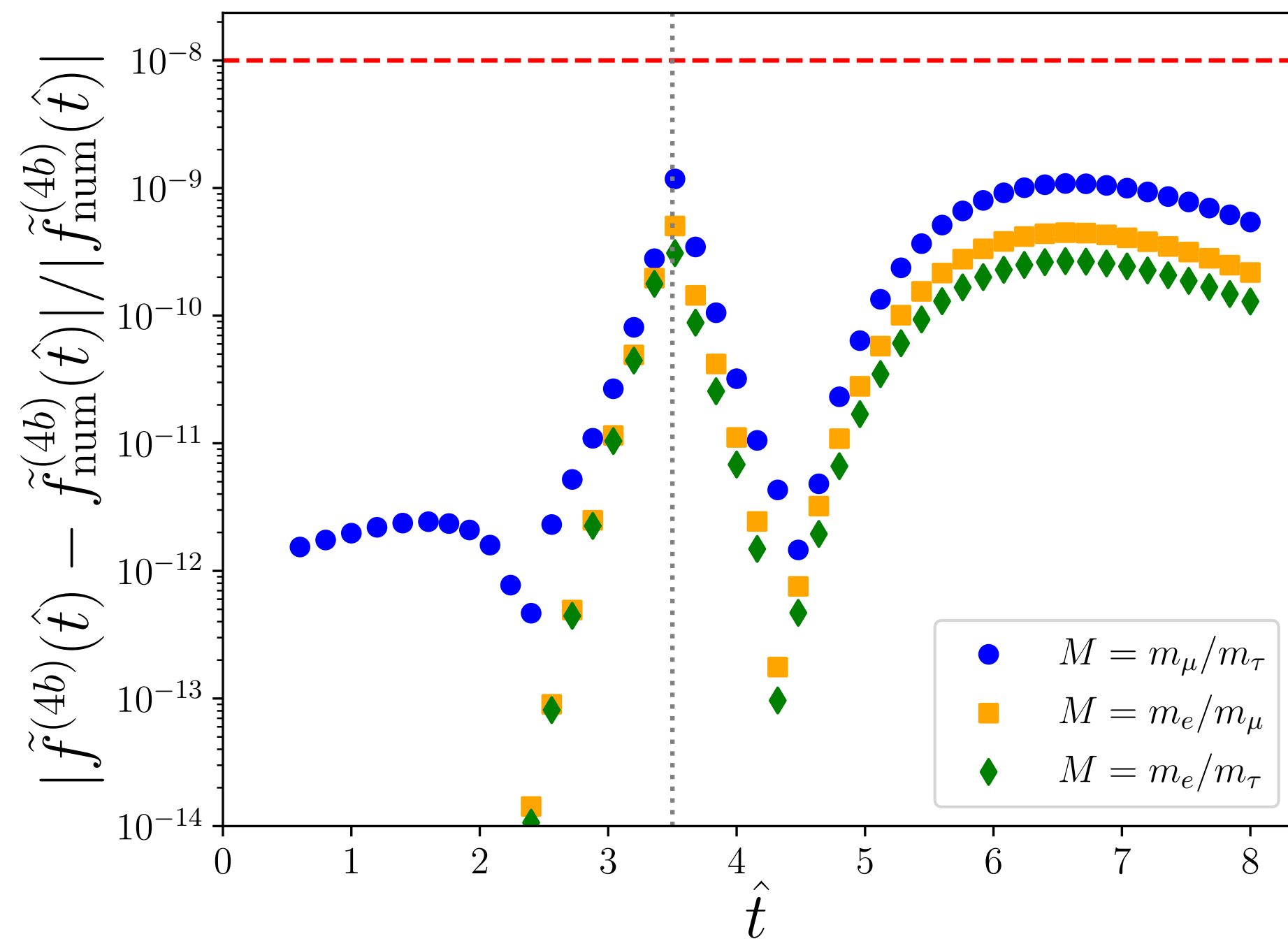
$$\frac{m_\ell^2}{16\pi^2} \tilde{f}_{\ell'}^{(4b)}(\hat{t}) \stackrel{\hat{t} \ll 1}{\equiv} \sum_{n \geq 4} \sum_{m \geq 0} \frac{\hat{t}^n}{n!} \left(\frac{m_{\ell'}}{m_\ell} \right)^m \left(e_{nm} + f_{nm} \pi^2 + g_{nm} (\gamma_E + \ln \hat{t}) + h_{nm} (\gamma_E + \ln \hat{t})^2 \right)$$

$$\frac{m_\ell^2}{16\pi^2} \tilde{f}_{\ell'}^{(4b)}(\hat{t}) \stackrel{\hat{t} \rightarrow \infty}{\equiv} \mathcal{D} \left(\hat{t}, \frac{m_{\ell'}}{m_\ell} \right) + \mathcal{S}(\textcolor{red}{m}_\ell \textcolor{red}{t}) + \mathcal{S}'(\textcolor{red}{m}_{\ell'} \textcolor{red}{t})$$

THREE REGIONS : $\hat{t} \ll 1 \ll 1/M$, $1 \ll \hat{t} \ll 1/M$ AND $1 \ll 1/M \ll \hat{t}$



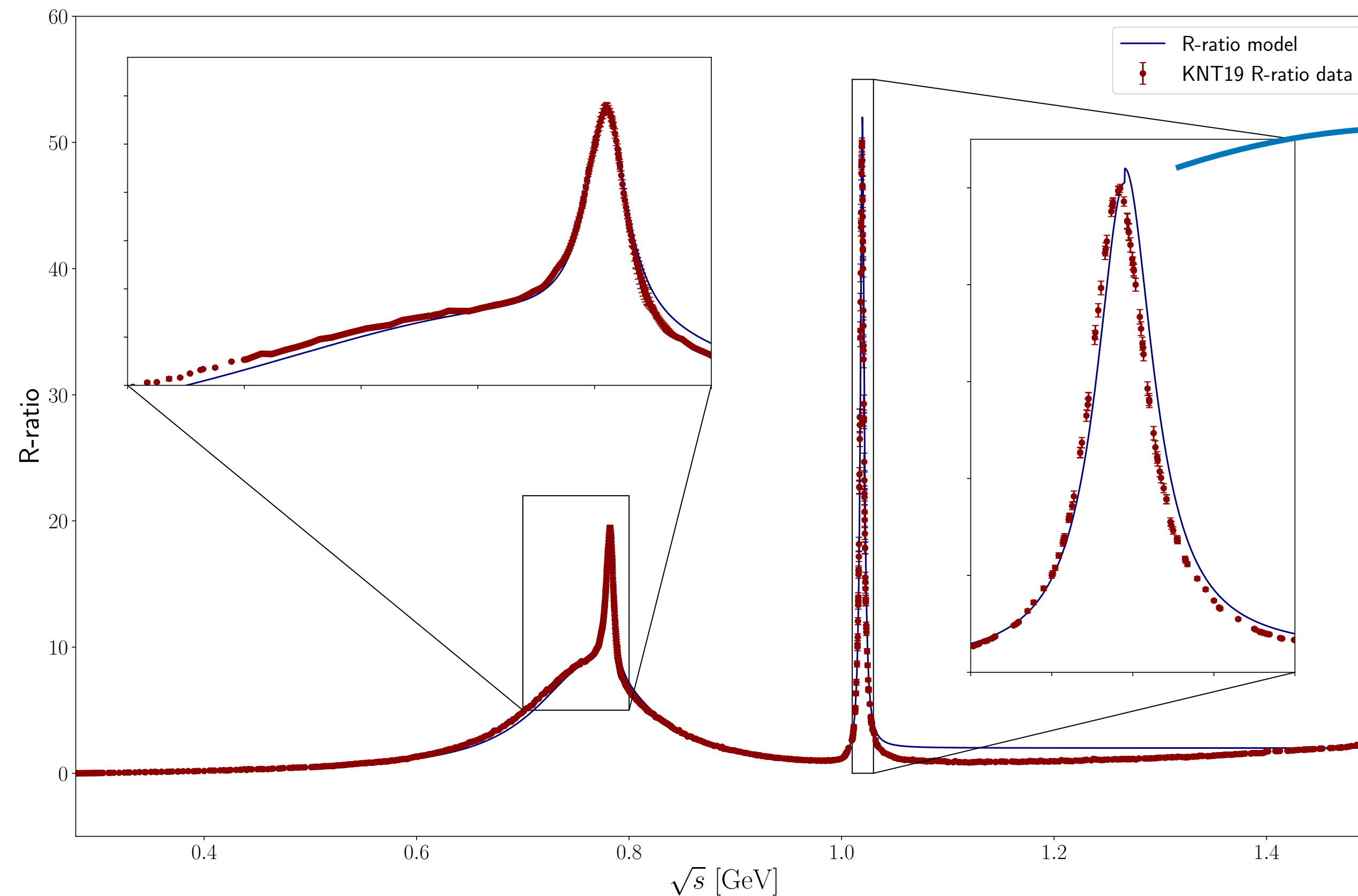
$$\frac{m_\ell^2}{8\pi^2} \tilde{f}^{(4b)}(\hat{t}) = 2 \int_0^\infty \frac{d\hat{\omega}}{\hat{\omega}} \hat{f}^{(2)}(\hat{\omega}^2) F_l(\omega; m_{\ell'}) \left[\hat{\omega}^2 \hat{t}^2 - 4 \sin^2 \frac{\hat{\omega} \hat{t}}{2} \right]$$



$$a_{\ell}^{\text{hvp}, (i)} = \left(\frac{\alpha}{\pi} \right)^{2+N_i} \int_0^{\infty} dq^2 f^{(i)}(\hat{q}^2) (\bar{\Pi}(q^2))^{m_i}$$

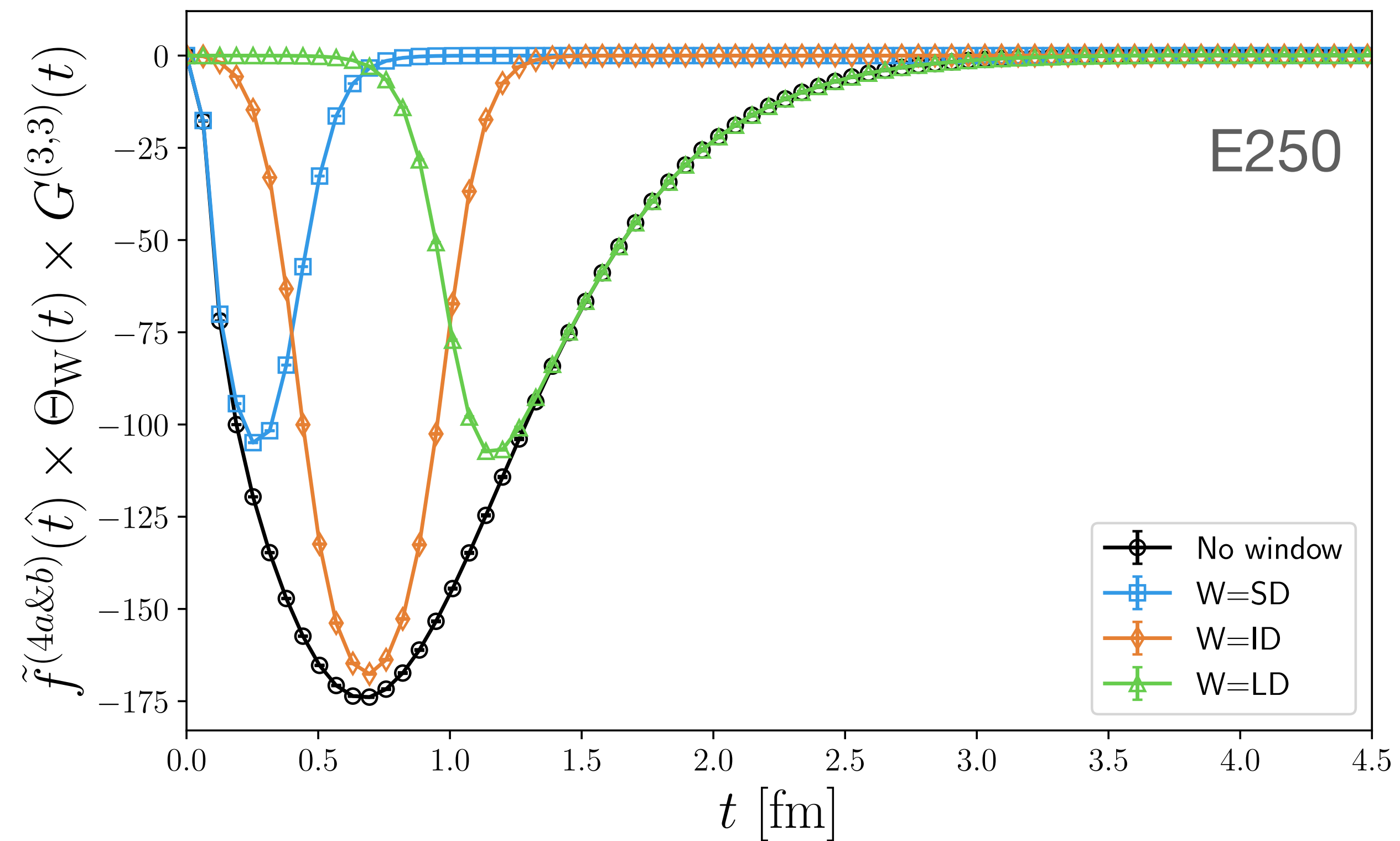
vs

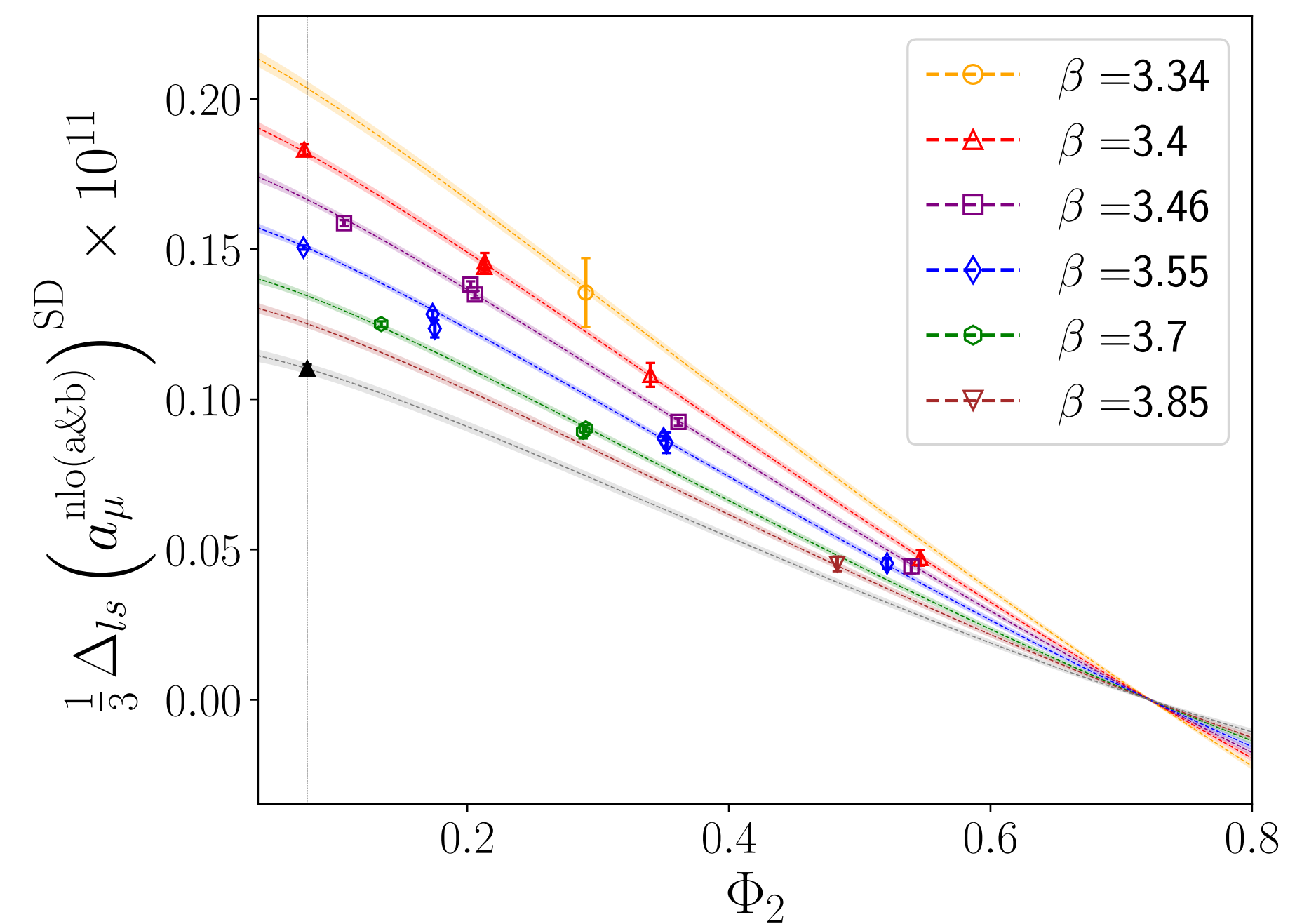
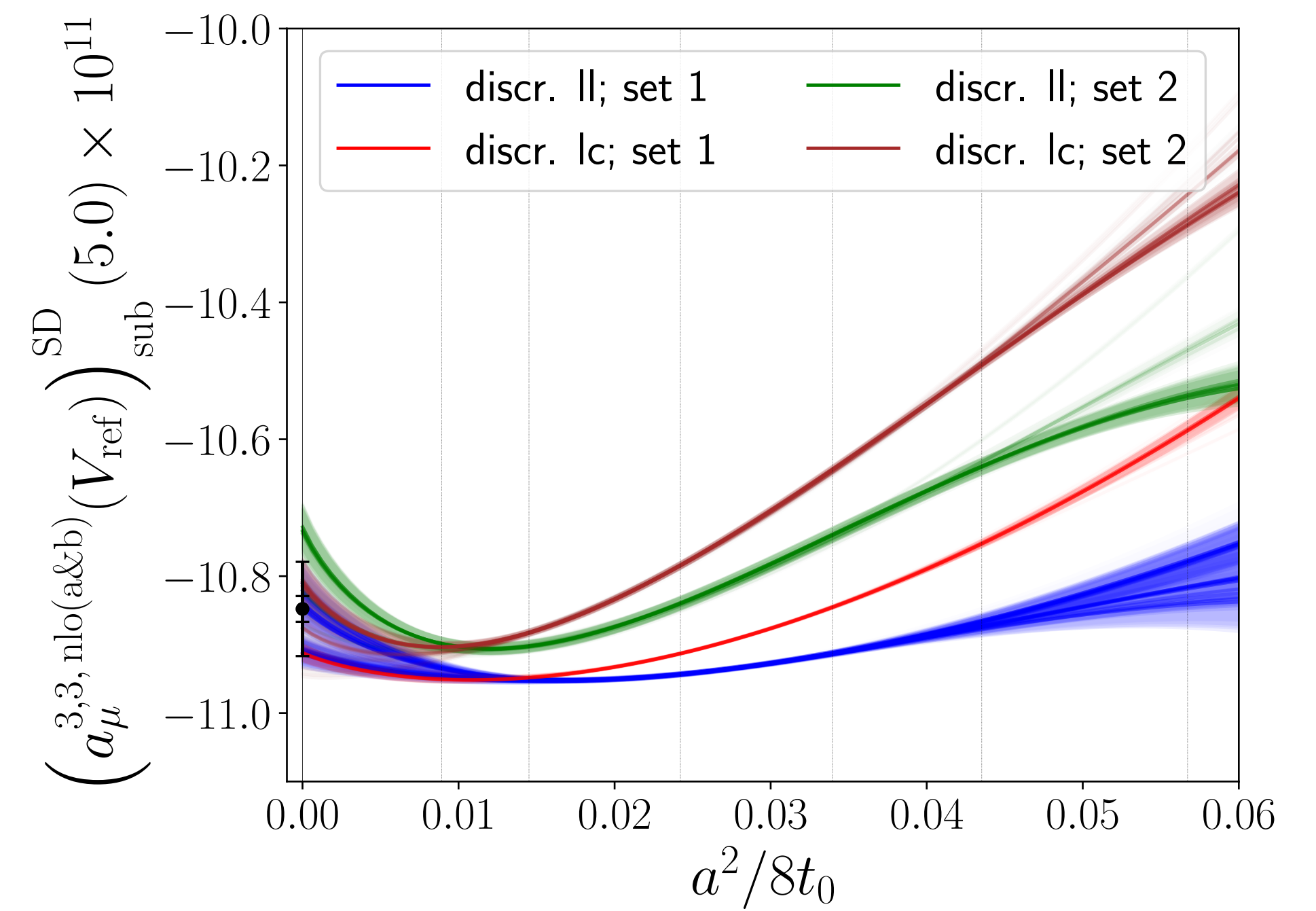
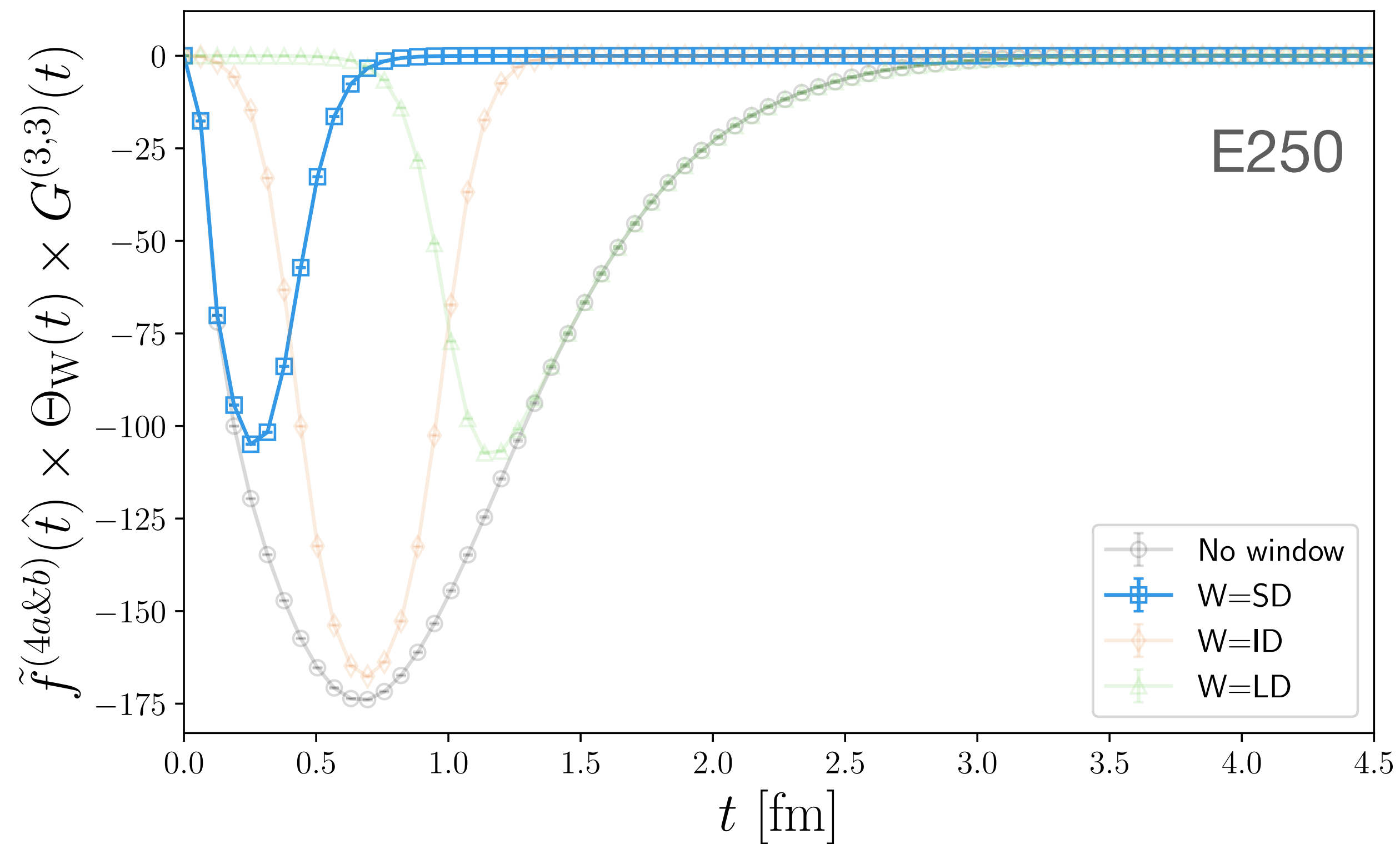
$$a_{\ell}^{\text{hvp}, (i)} = \left(\frac{\alpha}{\pi} \right)^{2+N_i} \int dt_1 \dots dt_{m_i} \tilde{f}^{(i)}(\hat{t}_1, \dots, \hat{t}_{m_i}) G(t_1) \times \dots \times G(t_{m_i})$$

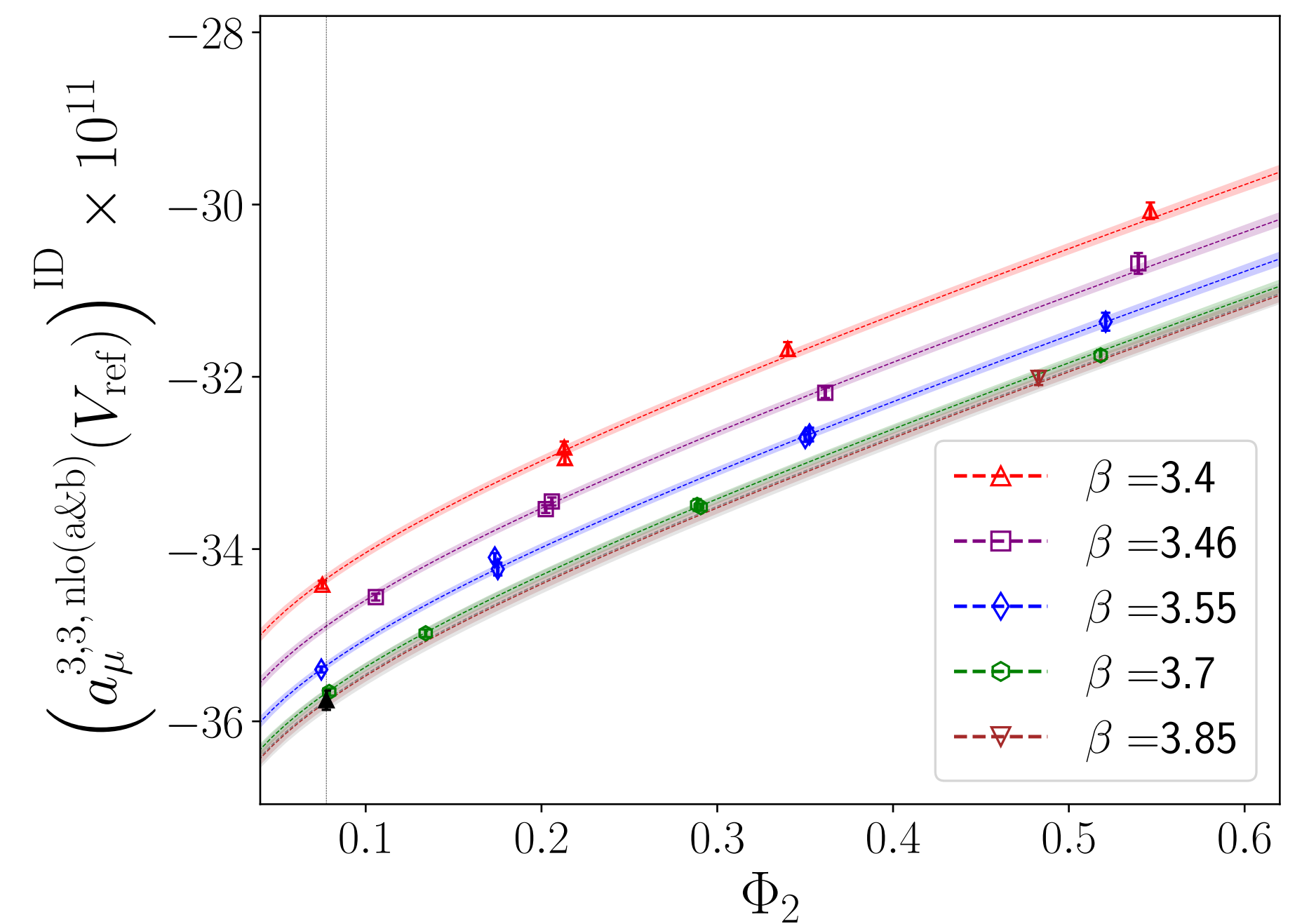
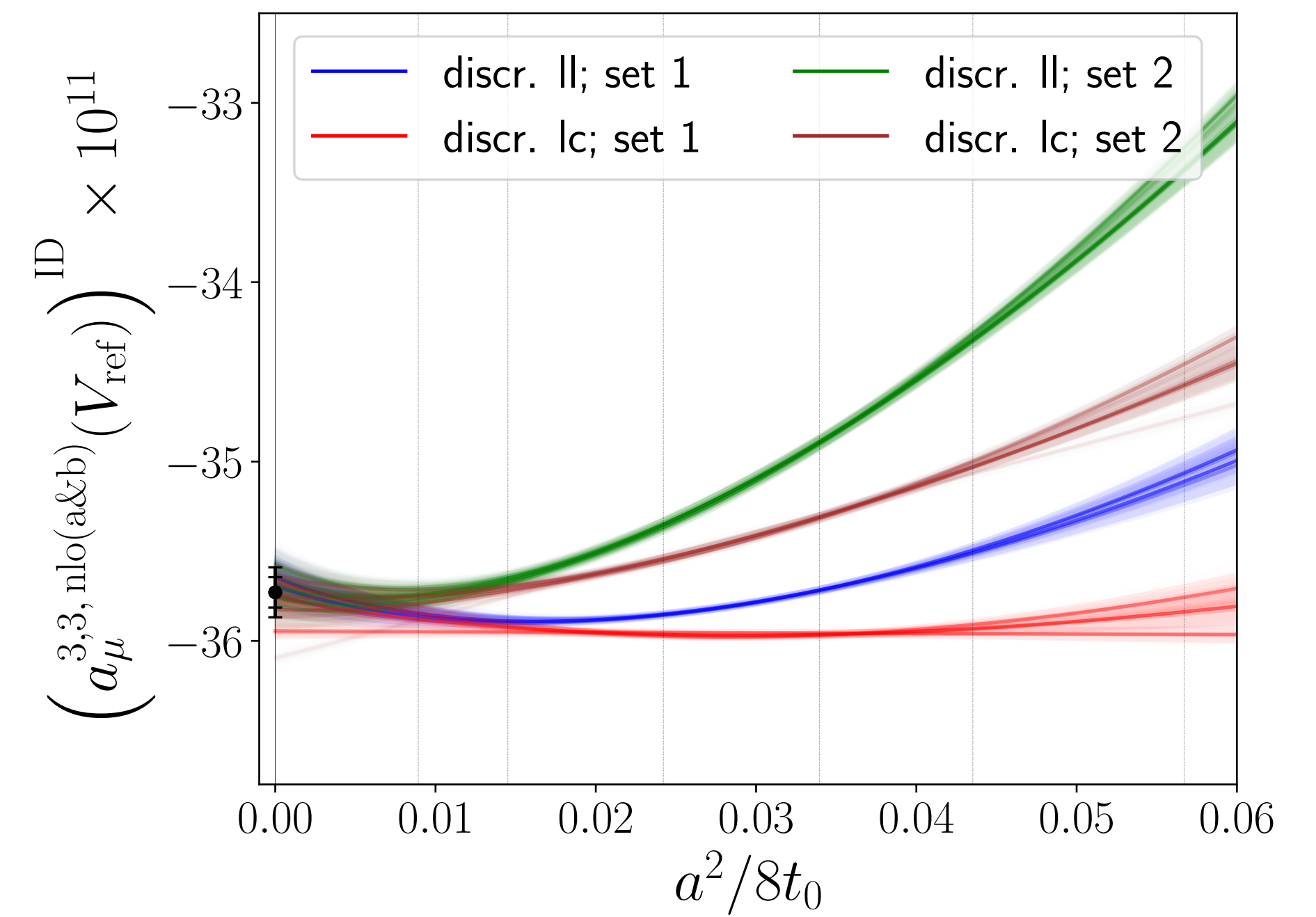
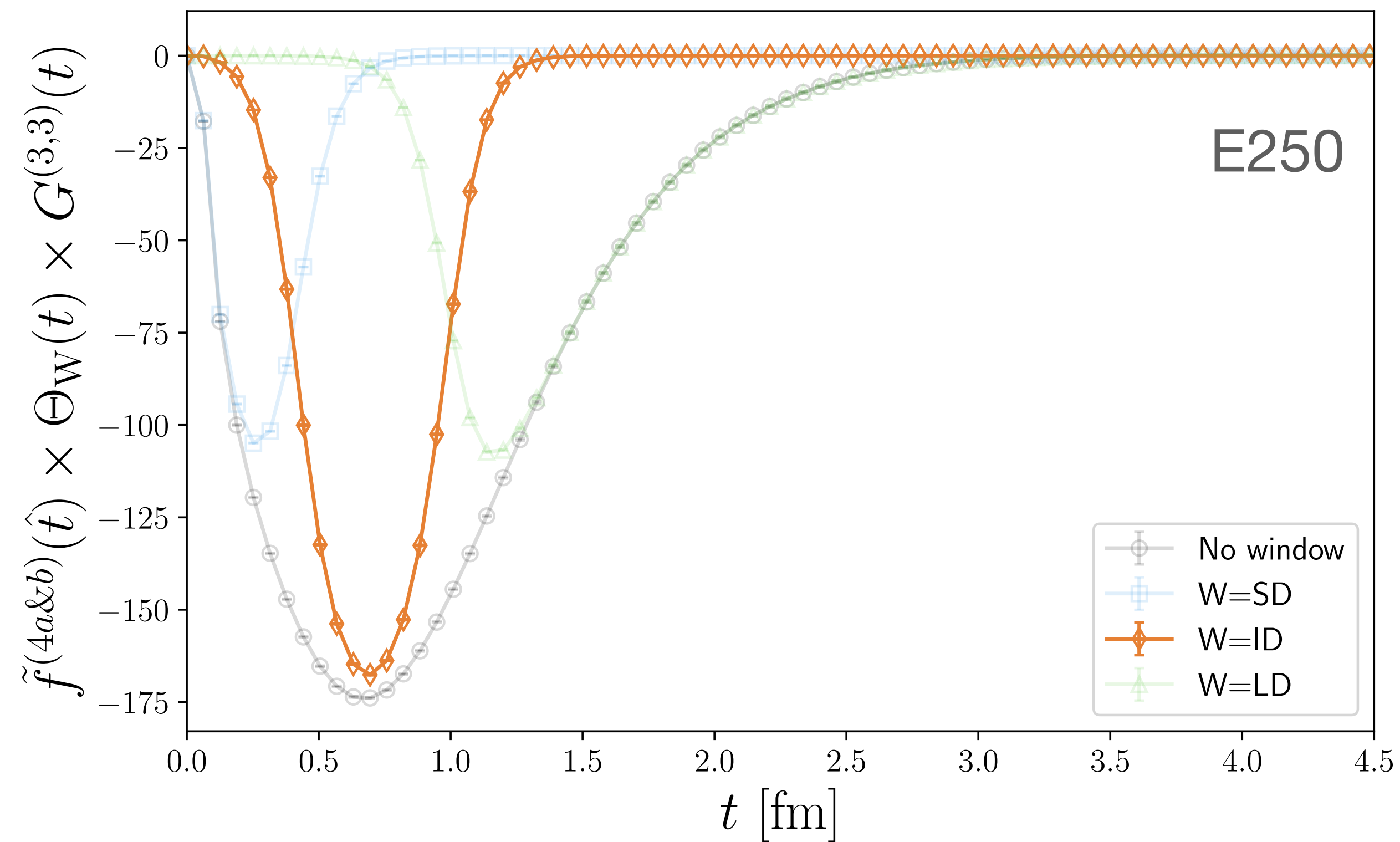


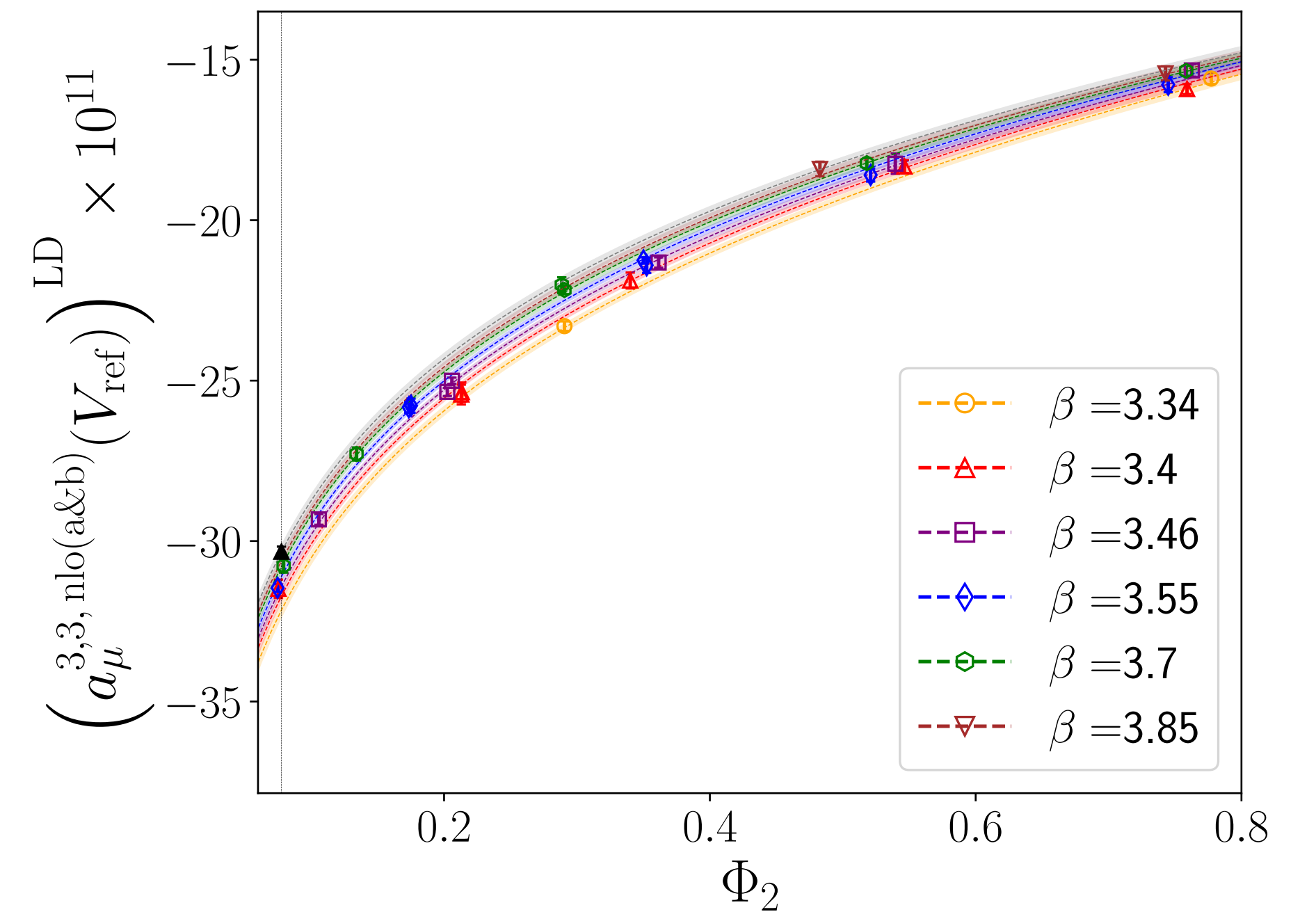
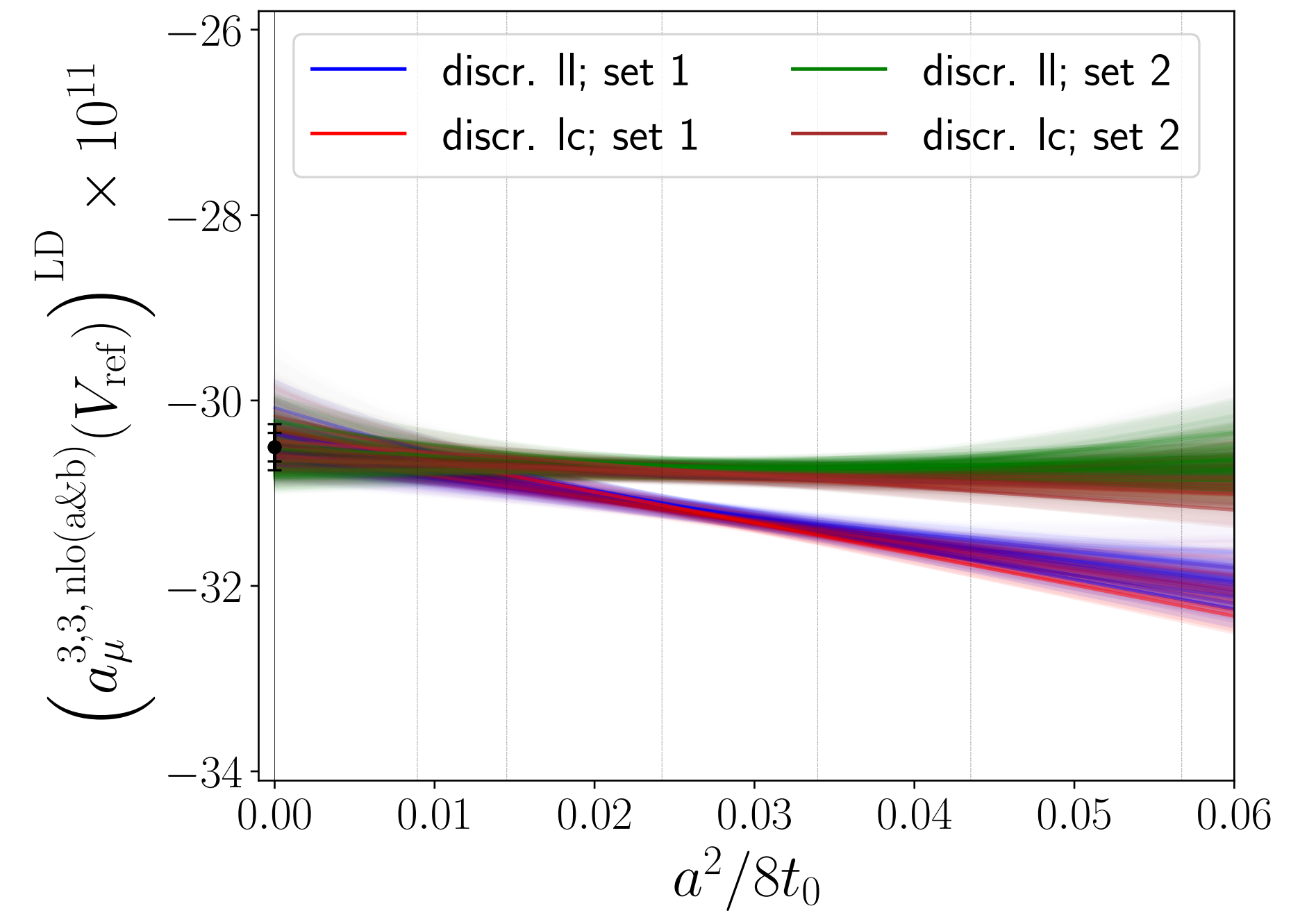
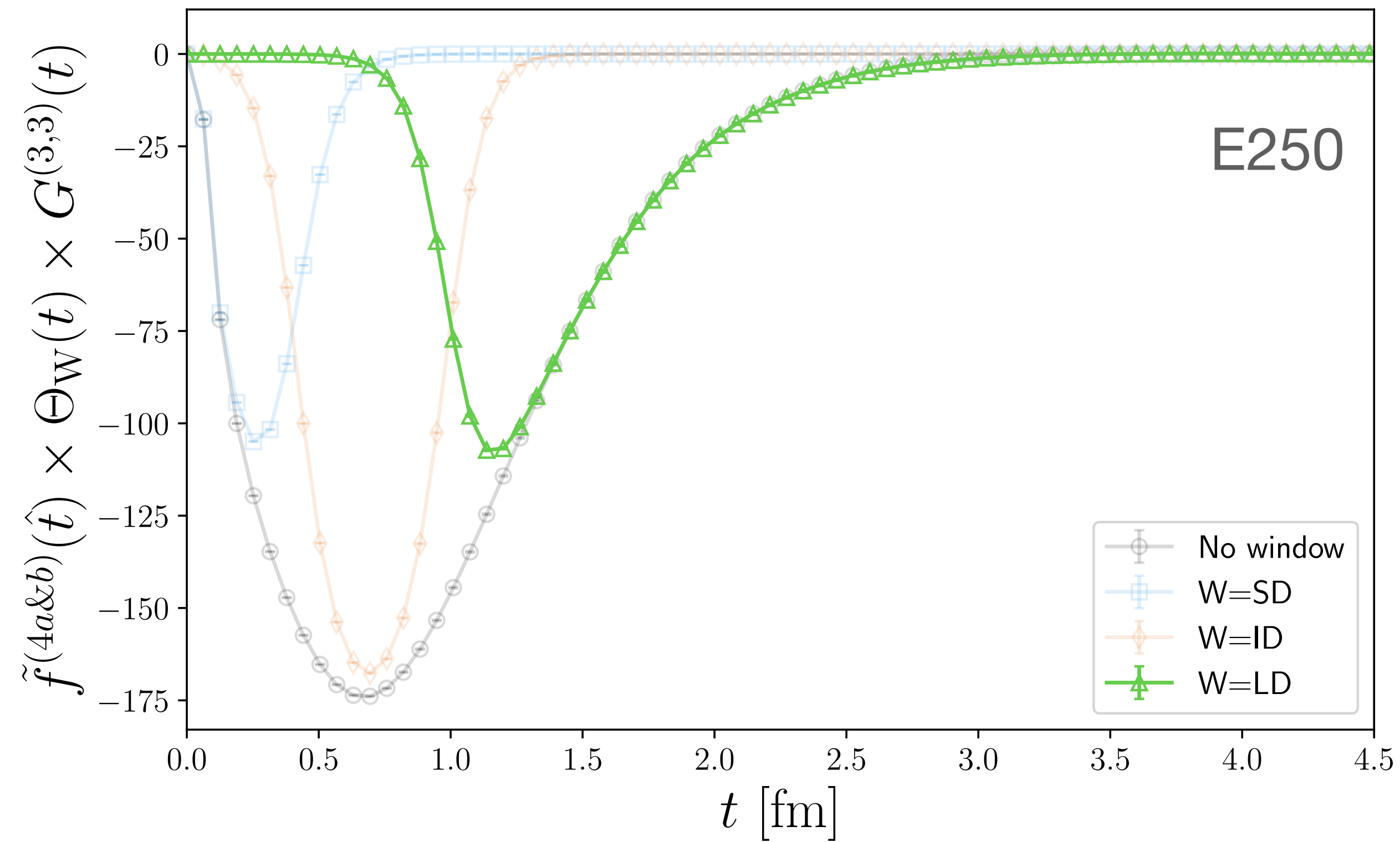
We relate the R -ratio to

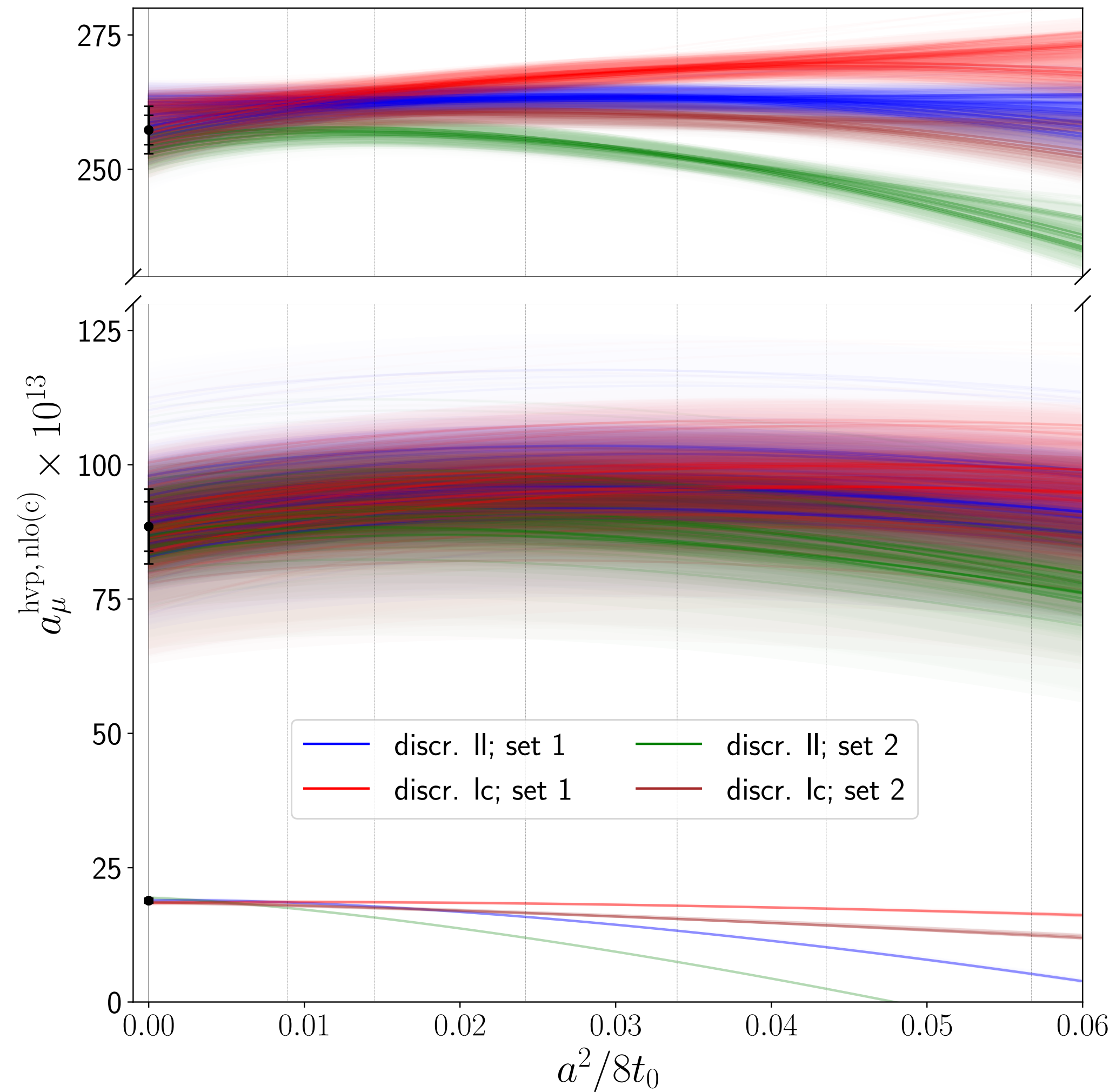
$\bar{\Pi}(\omega^2)$ and $G(t)$







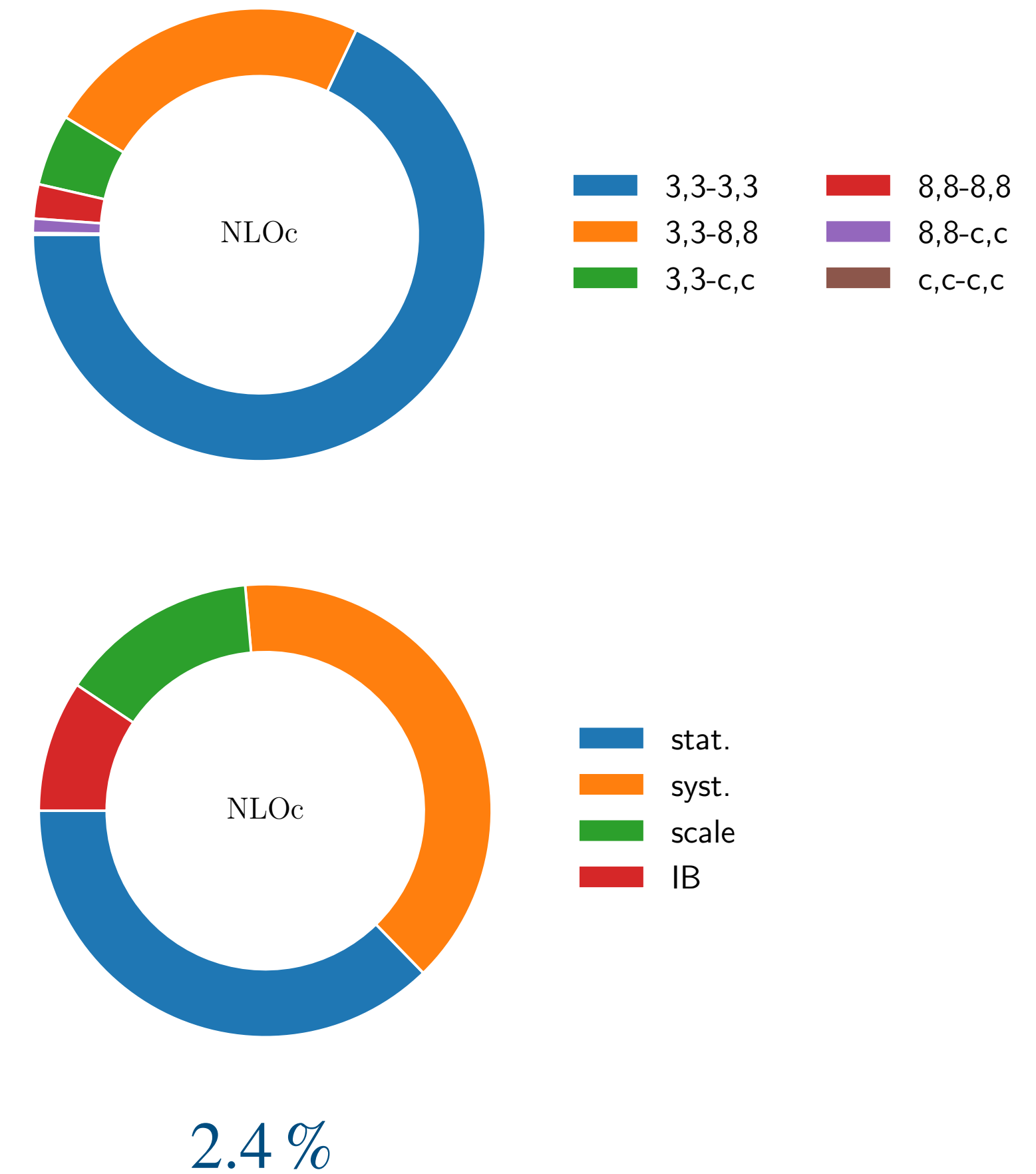




$$a_\mu^{3,3-3,3}$$

$$\frac{2}{3}a_\mu^{3,3-8,8}$$

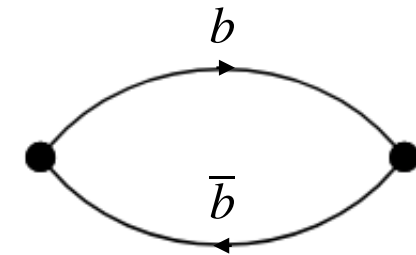
$$\frac{8}{9}a_\mu^{3,3-c,c}$$



Bottom quark effects

$m_b \gg 1/a$ for all CLS ensembles, we use pQCD

$$a_\mu^{b,b,(i)} = \left(\frac{\alpha m_\mu}{6\pi m_b} \right)^2 \left(\frac{\alpha}{\pi} \right)^{N_i} \int_1^\infty \frac{dz}{z^2} K^{(i)} \left(\frac{4m_b^2}{m_\mu^2} z \right) R_b(z)$$

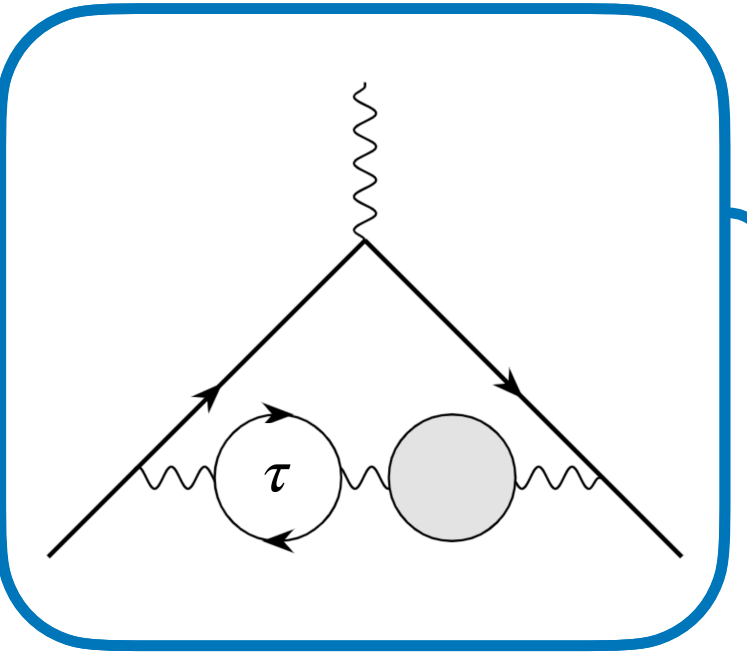


Tau-loop effects in NLO_b

We neglected NLO_b(τ)

We evaluate only in E250

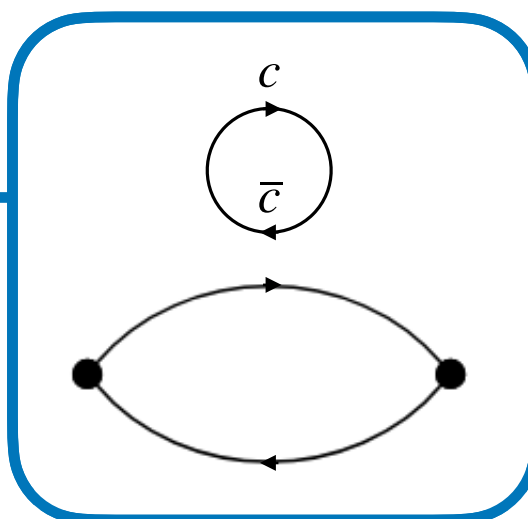
$$a_\mu^{\text{hvp, nlo}(\tau)} \approx \left(\frac{\alpha}{\pi} \right)^3 \int_0^\infty dt \tilde{f}_\tau^{(4b)}(\hat{t}) G_{\text{E250}}(t)$$



Charm sea-quark effects

$\mathcal{L}_E$ does not contain the charm quark

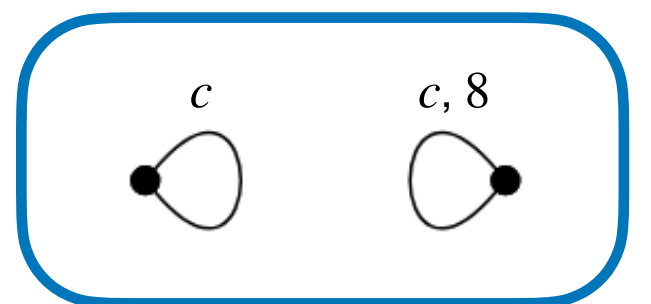
Direct and indirect effects from charm sea-quark loops are included in the error budget



Charm disconnected contributions

Caused by $\frac{2}{3\sqrt{3}} G_{\text{disc}}^{c8}$ and $\frac{4}{9} G_{\text{disc}}^{cc}$

Numerically irrelevant



- IsoQCD projects
 - Higher-order HVP in a_μ
 - Other lepton flavors
- Isospin breaking corrections

- SD+ID windows and the LD window have a similar contribution to the total variance (in isoQDD)
- **High precision** reached ($\lesssim 0.5\%$) in the isoQCD calculation

