The lattice scale setting procedure in the BMW/DMZ muon (g-2) calculation

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OUTLINE

- Scale setting using M_Ω
- New scale setting using Γ_π

The BMW-DMZ Collaboration

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Nature 653 (2026) 8114, 373-377 [arXiv:2407.10913]

Scale setting using M_Ω

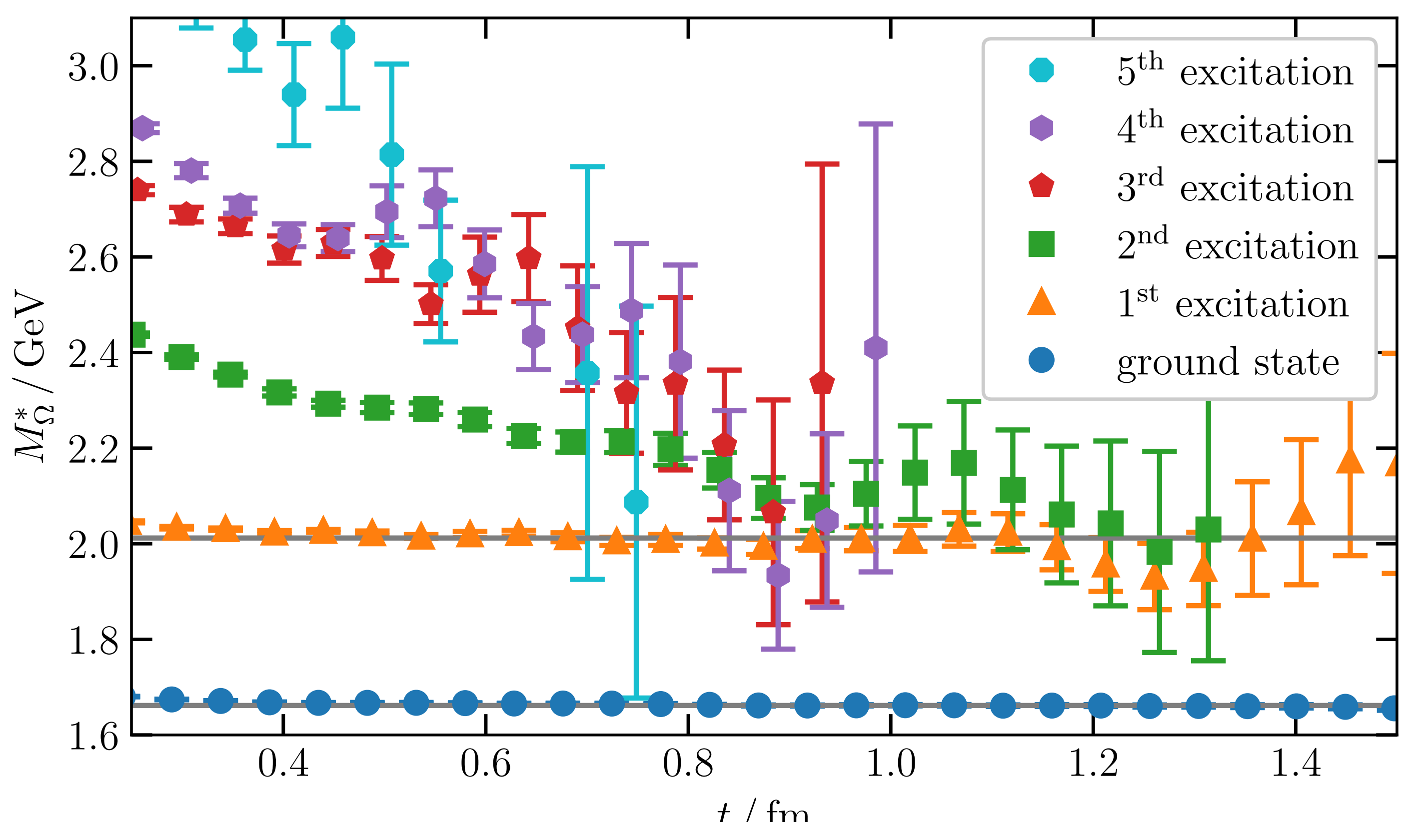
Introduction and Omega baryon mass

The calibration of lattice QCD simulations requires one dimensional quantity to set the lattice spacing: this can be either a physical (N, Ω baryon masses, $\mathcal{F}_\pi$ etc…) or a theory scale (r_0, w_0, M_{qq} etc…) $\mathcal{S}$

$$a = \left(\frac{(a^{d_\mathcal{S}} \mathcal{S})_{\text{lat}}}{\mathcal{S}_{\text{ref}}} \right)^{1/d_\mathcal{S}} \qquad \frac{\delta a_\mu^{\text{HVP}}}{a_\mu^{\text{HVP}}} \approx 2 \frac{\delta \mathcal{S}}{\mathcal{S}}$$

For a_μ^{HVP} scale setting required at few per-mille precision!

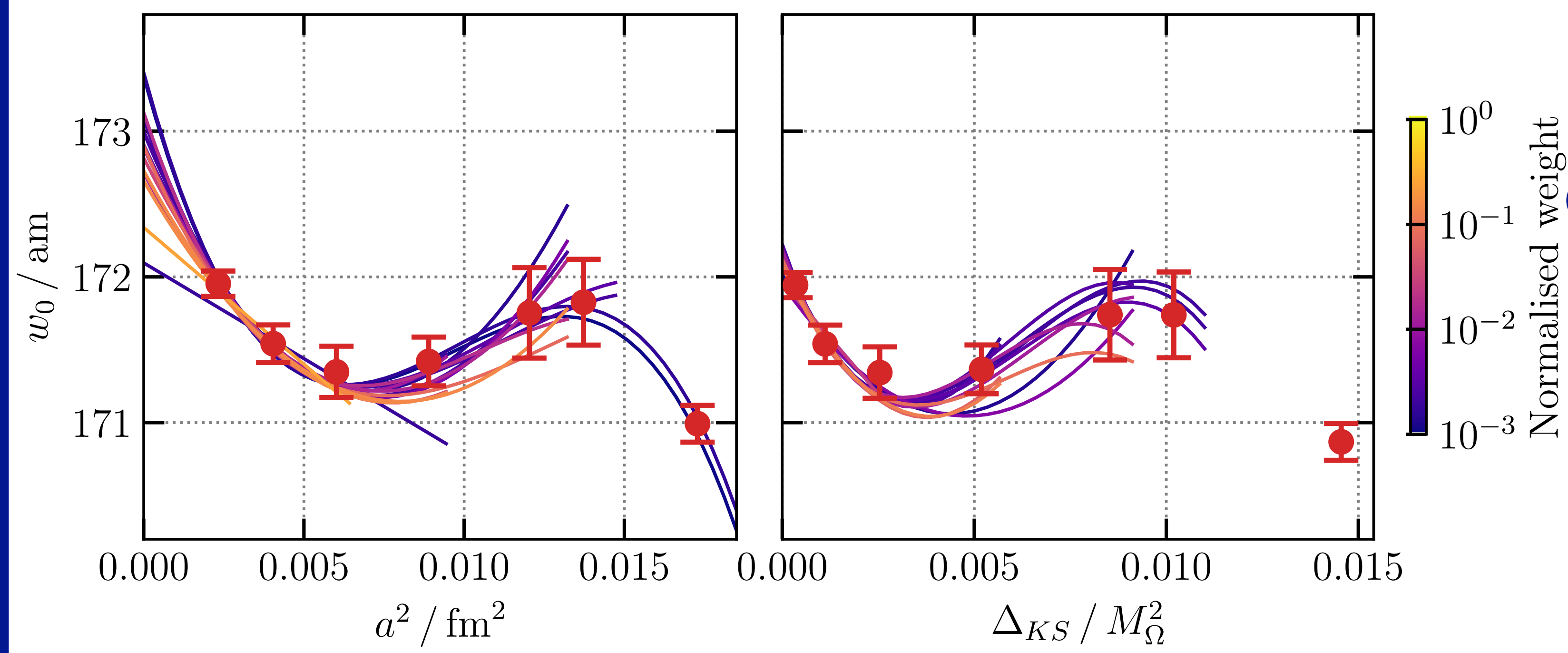
Solve the Generalized Eigenvalue Problem (3 op, point and smeared sources)



- Over 30 000 gauge configurations
- 10's of millions measurements
- States consistent with resonances and quark model

Determination of the gradient-flow scale w_0

$$w_0 M_\Omega = A + B\ M_\Omega^{-2} M_{ud}^2 + C\ M_\Omega^{-2}(M_{us}^2 + M_{ds}^2 - M_{ud}^2)/2 + E\ e_v^2 + F\ e_v e_s + G\ e_s^2$$

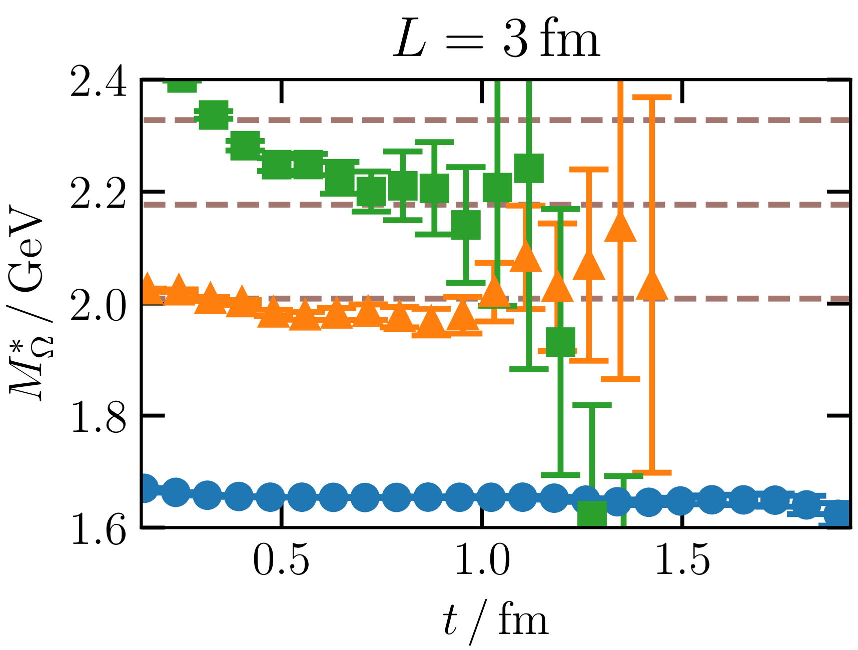


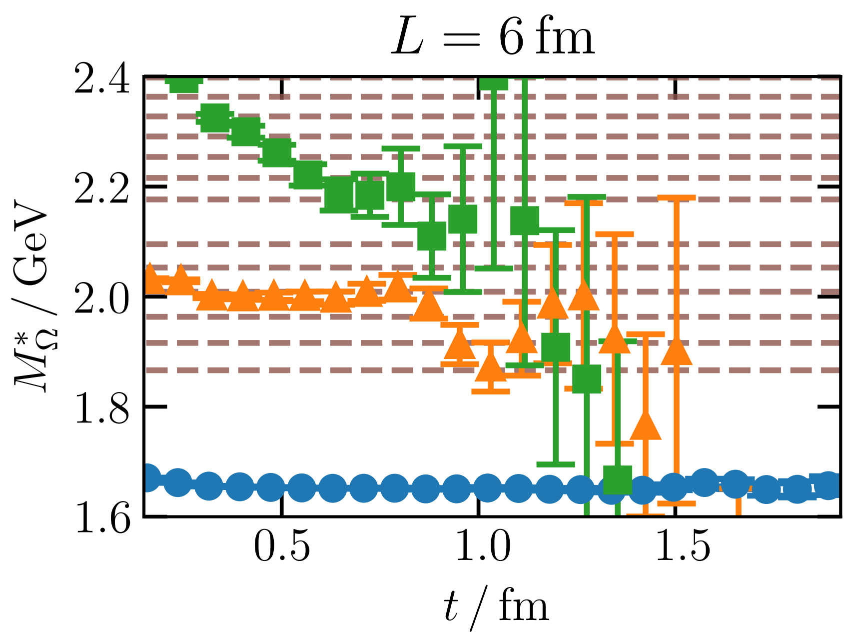
New ensemble with $a = 0.048$ fm

Cubic polynomials in a^2 and $\alpha_s^\gamma a^2$

$$[w_0]_{\rm qcd+qed} = 0.17245(22)(46)[51]\ \text{fm}$$

The determination of the lattice scale through M_Ω at 0.2% level may be problematic: in GEVP studies potential <u>contaminations from scattering states</u> should be considered





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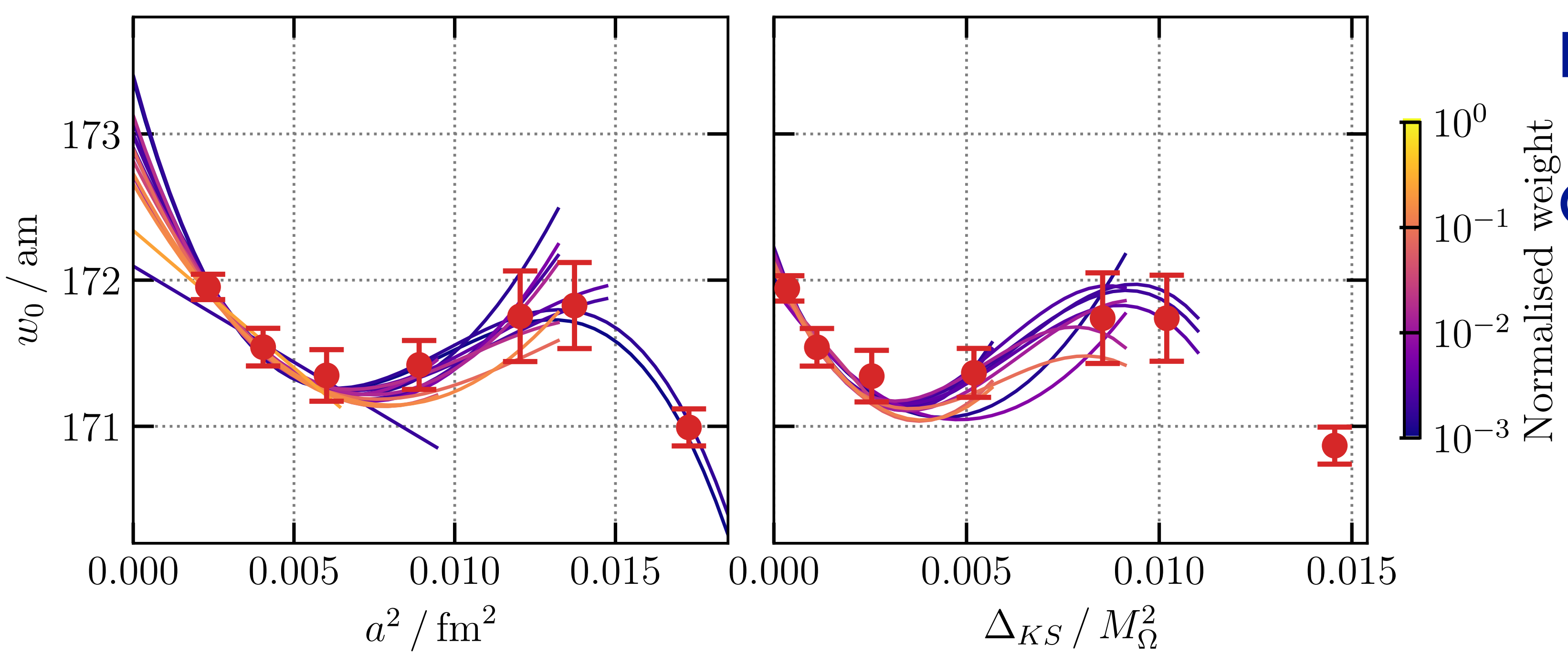

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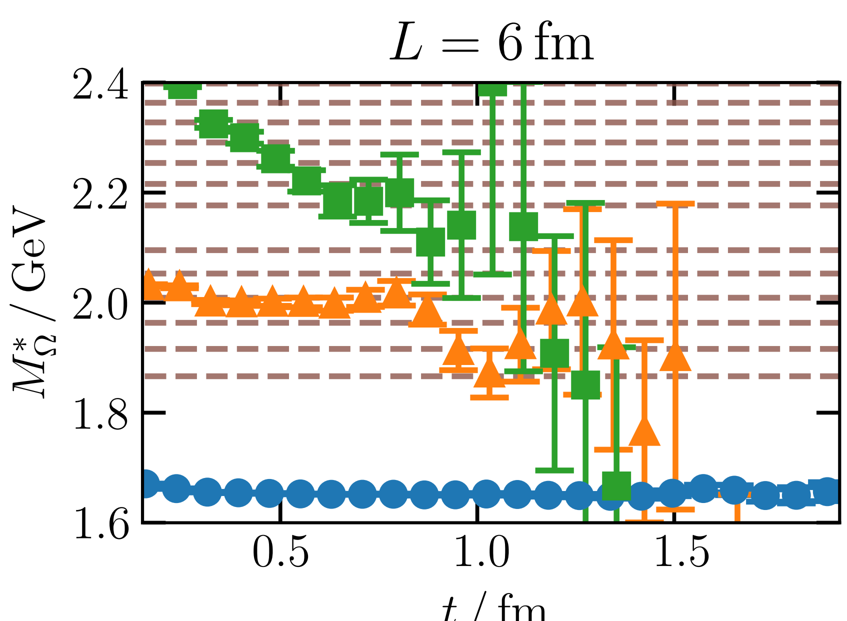
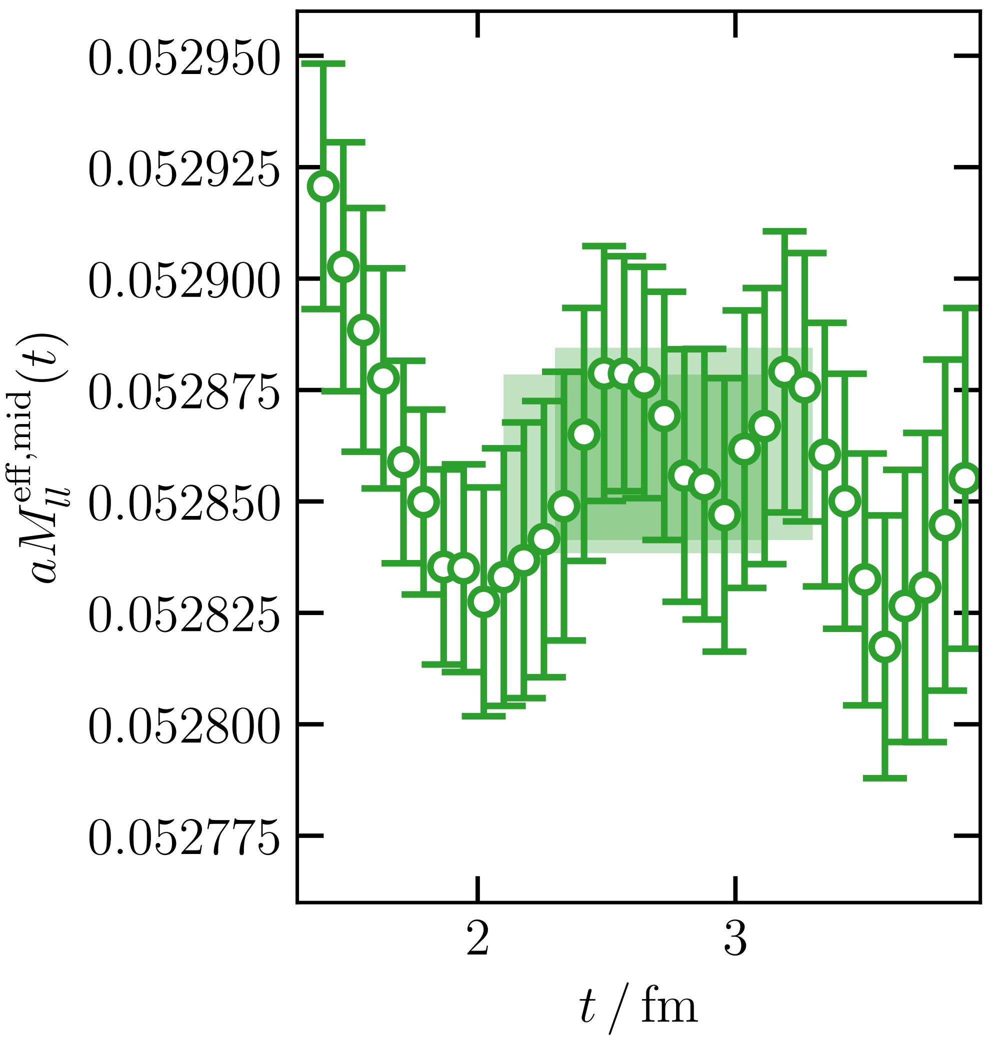


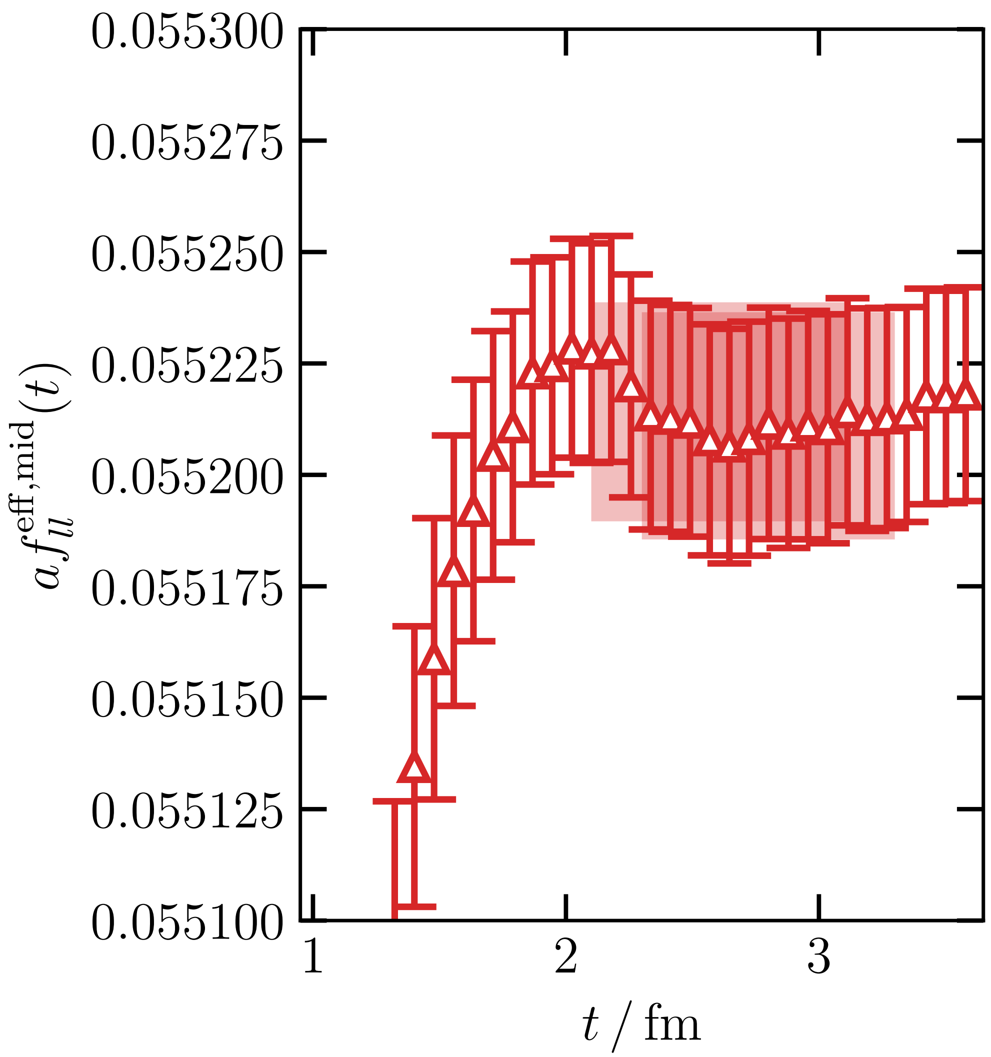
We searched in vain for signs of these contaminations by changing smearing, volume and performing multi-state fits.

No concrete bound on uncertainty

Scale setting using Γ_π

Pion mass and decay constant





$$M_{ll}^{\text{eff, loc}}(t) = \frac{1}{\Delta}\cosh^{-1}\frac{G(t+\Delta)+G(t-\Delta)}{2G(t)}$$

$$M_{ll}^{\text{eff, mid}}(t) = -\frac{1}{2\Delta}\left[\cosh^{-1}\frac{G(t+\Delta)}{G(T/2)} - \cosh^{-1}\frac{G(t-\Delta)}{G(T/2)}\right]$$

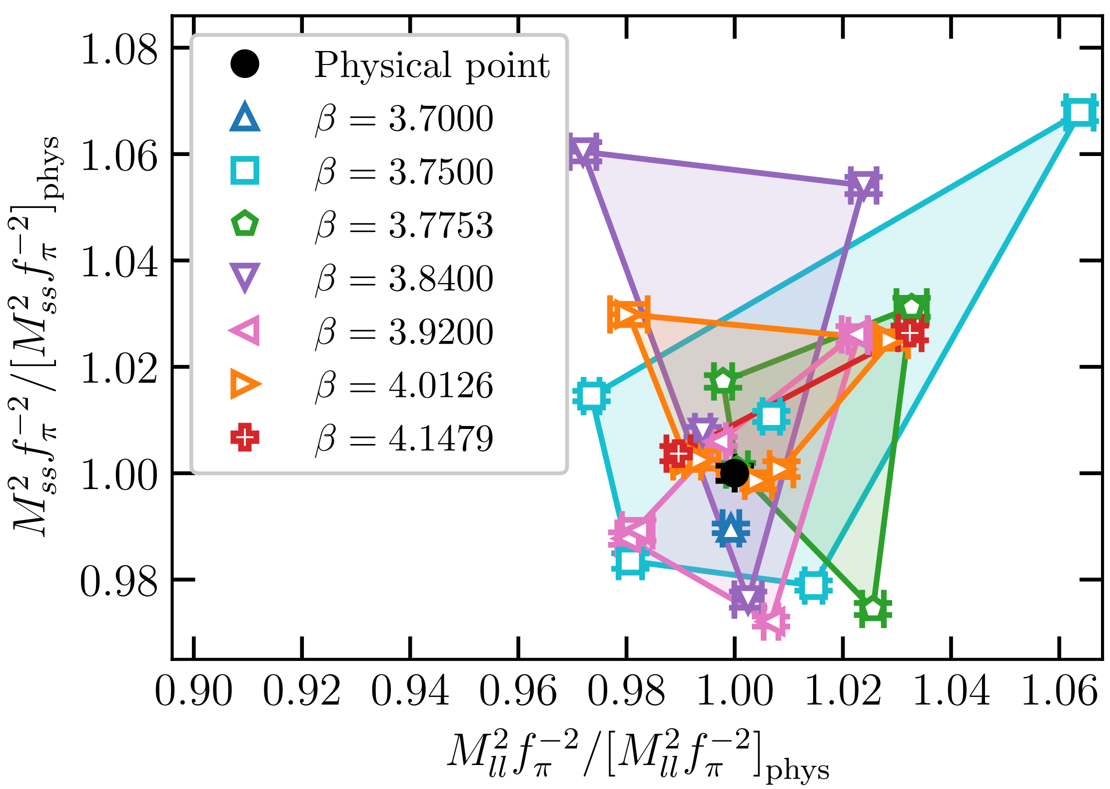
$$f_{ll}^{\text{eff}}(t) = \sqrt{\frac{8m_l^2 G(t)}{M_{ll}^{\text{eff}}(t)^3}\,\frac{e^{M_{ll}^{\text{eff}}(t)\frac{T}{2}}\left(1-e^{-M_{ll}^{\text{eff}}(t)T}\right)}{\cosh\left(M_{ll}^{\text{eff}}(t-\frac{T}{2})\right)}}$$

Effective decay constant from PP correlator shows clear plateau

Radiative corrections

The use of Γ_π to set the lattice scale requires accounting for radiative corrections

$$F_{ud}^2 = \Gamma(\pi \to \mu\bar{\nu}_\mu[\gamma]) \left[\frac{G_F^2}{8\pi}|V_{ud}|^2 M_{ud} m_\mu^2 \left(1 - m_\mu^2/M_{ud}^2\right)\right]^{-1}$$

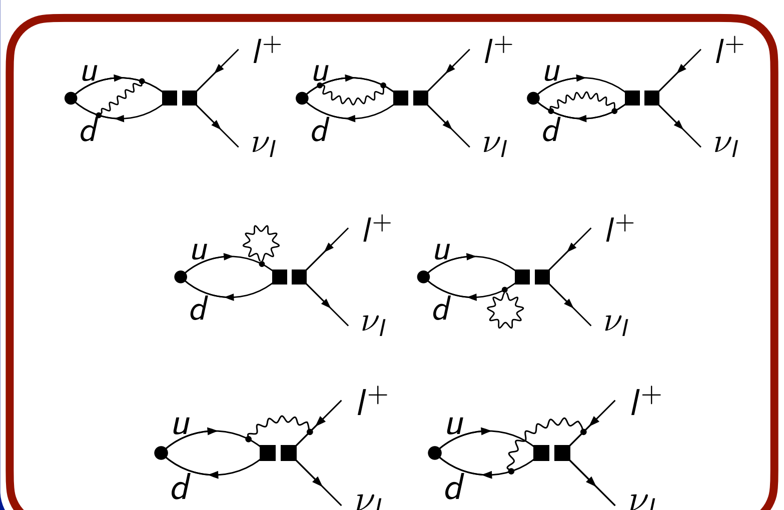


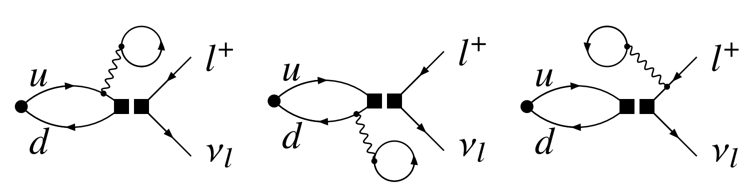
- QCD tree level Symanzik Action;
- $N_f = 2+1+1$ Staggered fermions;
- 4 levels of stout smearing $\rho = 0.125$;
- $L \approx 6$ fm;

$$[O]_{\rm qcd+qed} = [O]_{\rm qcd} + \boxed{[O]_{\rm qed,vv}} + [O]_{\rm qed,vs} + \boxed{[O]_{\rm qed,ss}}$$

taken from electroquenched RM123S calculation Di Carlo *et al.* 2019

computed using our fermion determinant derivatives

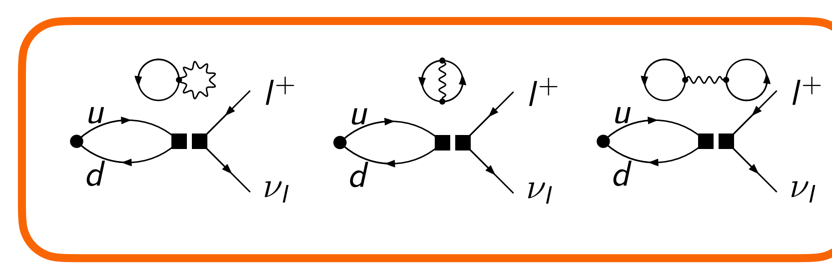




$SU(3)$ flavor suppression

XPT $O(e^2p^4)$: -0.01% corr. on iso Γ

M. Knecht *et al.* 2000; J. Bijnens *et al.* 2007



leading EUQ effects

Radiative corrections

We adopt the qed,vv contribution from Di Carlo *et al.* 2019 in the GRS scheme

$$[F_{ud}]_{\rm qed,vv}/[F_{ud}]_{\rm qcd} = (1+\delta R_\pi)^{1/2} - 1 \quad \text{with} \quad \delta R_\pi = 0.0150(18)$$

DG *et al.* 2017 provides the QCD and QED decompositions of pion and kaon masses in the same scheme, in particular $[O]^{\rm phys}_{\rm qcd+qed} - [O]_{\rm qed,vv}$ is given

$$[M_{ud}]_{\rm qcd++} = 135.0(2)\ \mathrm{MeV}$$
$$[\tfrac{1}{2}(M_{us}+M_{ds})]_{\rm qcd++} = 494.6(1)\ \mathrm{MeV}$$

$$[O]_{\rm qcd++} \equiv [O]_{\rm qcd} + [O]_{\rm qed,vs} + [O]_{\rm qed,ss}$$

Adopting the physical value $[F_{ud}]^{\rm phys}_{\rm qcd+qed} = 131.711(45)\ \mathrm{MeV}$ one gets

$$[F_{ud}]_{\rm qcd++} = [F_{ud}]^{\rm phys}_{\rm qcd+qed}/(1+\delta R_\pi)^{1/2} = 130.73(12)\ \mathrm{MeV}$$

Note: In EQ approx. adopted in Di Carlo *et al.* valence QED corrections do not change the lattice spacing from its pure QCD value at fixed β

Determination of the gradient-flow scale w_0

- Up to HO corrections we adopt the following parameterization

$$w_0 F_{ud} = A \ + B\ F_{ud}^{-2} M_{ud}^2 + C\ F_{ud}^{-2}(M_{us}^2 + M_{ds}^2 - M_{ud}^2)/2 + E\ e_v^2 + F\ e_v e_s + G\ e_s^2$$

All quantities (after renormalization) are independent, finite observables in full theory, thus the fit parameters are finite and independent of any scheme

- From iso-QCD we get A, B, C: $\quad w_0 f_{ll} = A + B\ f_{ll}^{-2} M_{ll}^2 + C\ f_{ll}^{-2}(M_{ls}^2 - \frac{1}{2} M_{ll}^2)$

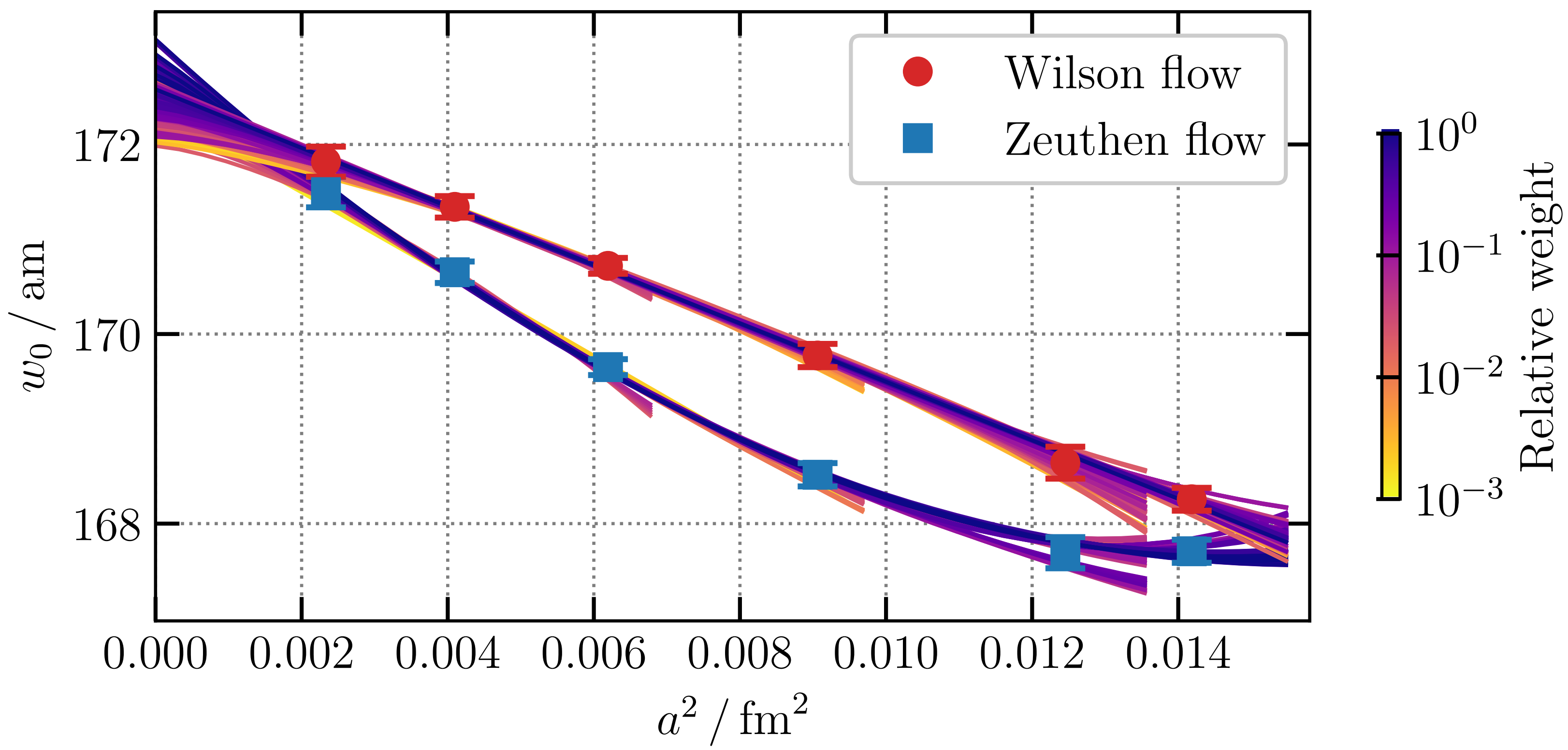
- The qed,ss contribution, up to HO effects, is computed via

$$G = \partial_{02} \left(w_0 F_{ud} - B\ F_{ud}^{-2} M_{ud}^2 - C\ F_{ud}^{-2}(M_{us}^2 + M_{ds}^2 - M_{ud}^2)/2 \right)$$

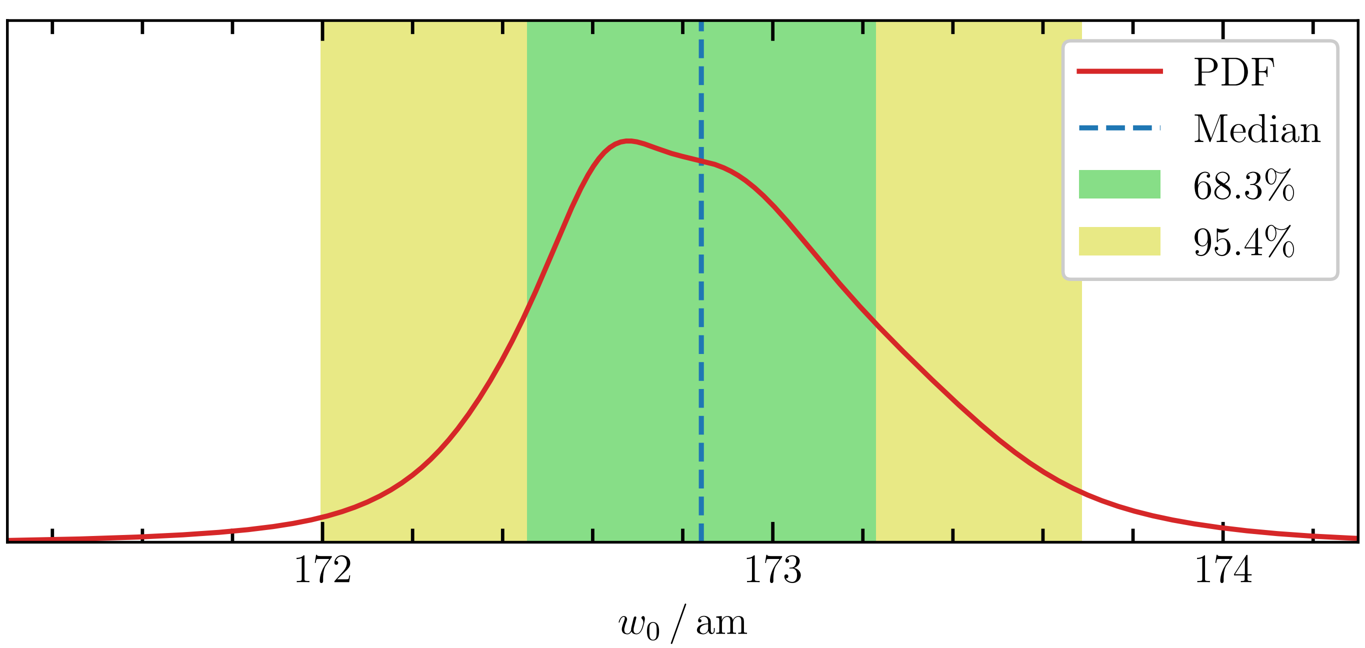
- Putting together $[w_0 F_{ud}]_{\rm qcd+qed} = [w_0 F_{ud}]_{\rm qcd++} + [w_0 F_{ud}]_{\rm qed,vv}$ $\quad \left[\frac{[w_0 F_{ud}]_{\rm qed,vv}}{[w_0 F_{ud}]_{\rm qcd}} = (1 + \delta R_\pi)^{1/2} - 1 \right]$

$$\boxed{[w_0]_{\rm qcd+qed} = [w_0 F_{ud}]_{\rm qcd++} / [F_{ud}]_{\rm qcd++}}$$

Determination of the gradient-flow scale w_0



Number of fits	12672	
Fits with $P > 0.1$	74%	
Median	172.84	
Total error	0.42	0.24%
Statistical error	0.34	0.20%
Systematic error	0.25	0.14%
Pseudoscalar fits	0.15	0.09%
Finite volume XPT	0.03	0.02%
Type of flow (Ze/Wi)	0.14	0.08%
Lattice spacing cut	0.04	0.02%
Fit polynomial order	0.11	0.06%
Log corrections γ	0.07	0.04%



$$[M_{ud}]_{\text{qcd,FLAG}} = 135.0 \text{ MeV}$$

$$[\tfrac{1}{2}(M_{us} + M_{ds})]_{\text{qcd,FLAG}} = 494.6 \text{ MeV}$$

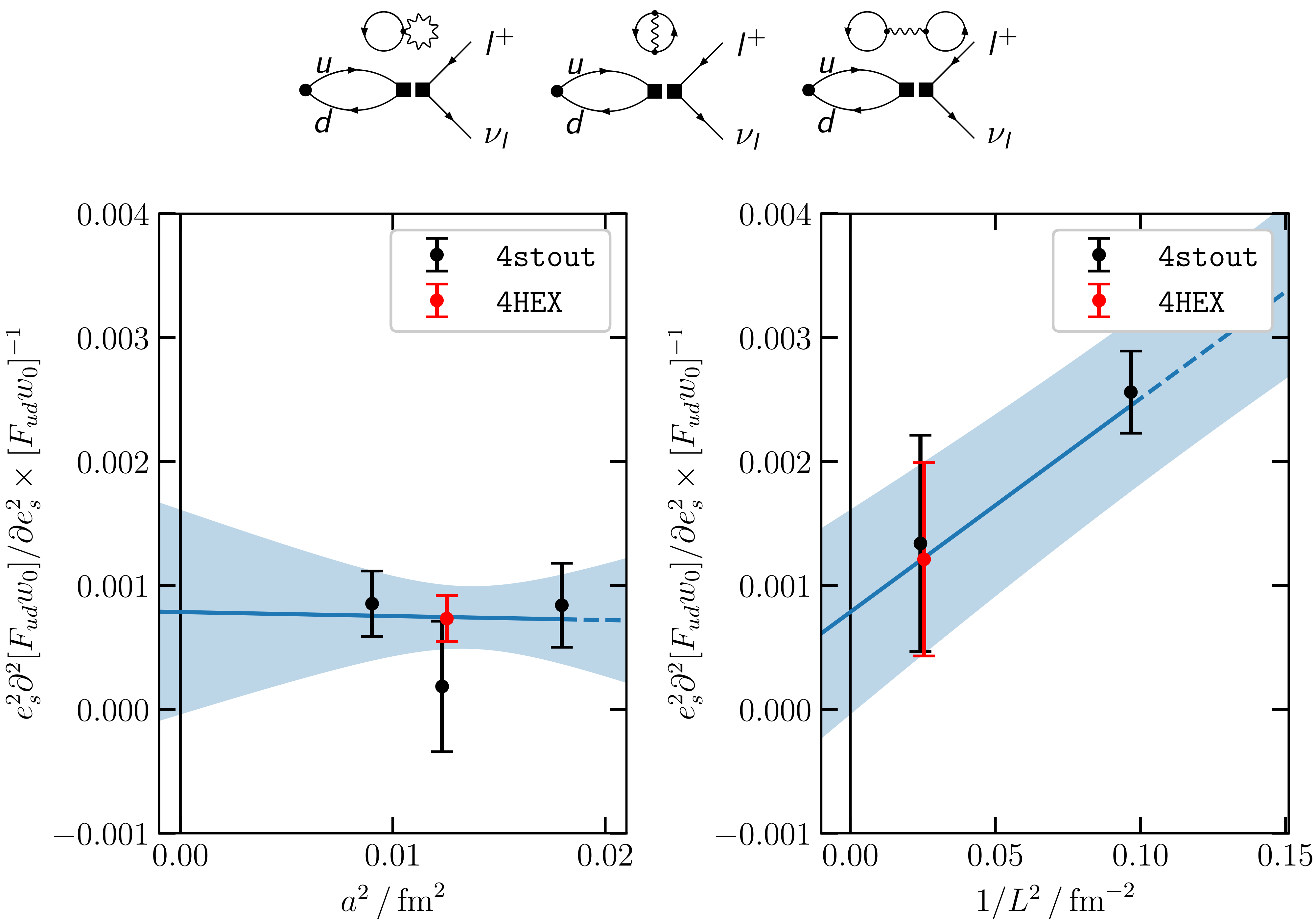
$$[F_{ud}]_{\text{qcd,FLAG}} = 130.5 \text{ MeV}$$

$$\boxed{[w_0]_{\text{qcd,FLAG}} = 0.17284(34)(25)[42] \text{ fm}}$$

w/o $a = 0.048$ fm $\quad 0.17279(37)(40)[54]$ fm

22% unc. red.

Sea-Sea contribution

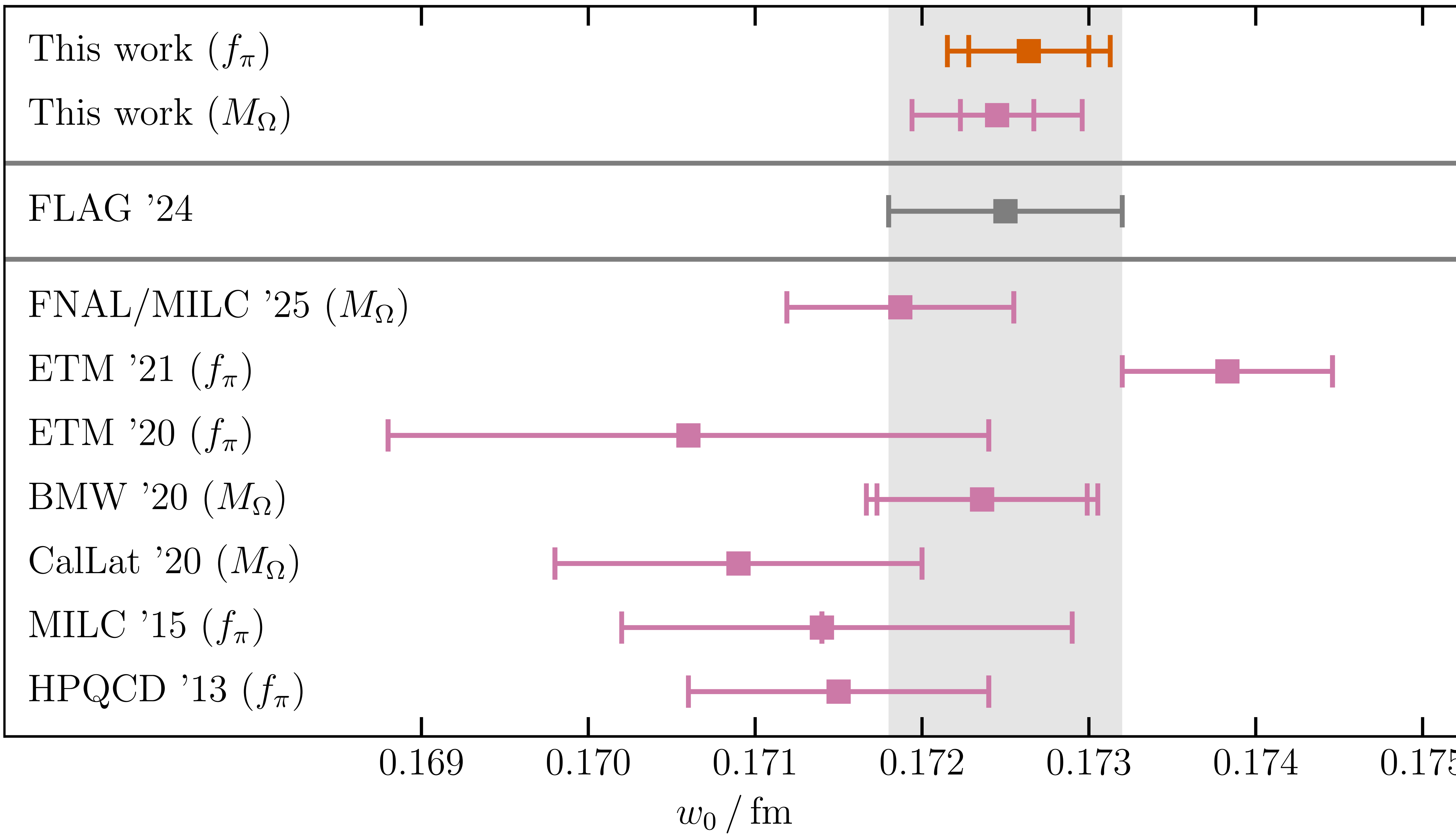


$$[w_0]_{\rm qcd+qed} = 0.17264(36)(33)[48]\ \text{fm}$$

$$(33) = (27)_{\rm qcd}(18)_{\rm qvv}(9)_{\rm qss}(6)_{V_{ud}}(3)_{\rm exp}$$

Summary comparison

All our results have been previously blinded



Conclusions and future perspectives

- The Ω baryon mass determination is potentially affected by multi-hadron excited state contaminations and, when w_0 is set through M_Ω, it shows a wavering continuum limit behavior for our staggered action

- Using the pion decay rate we can reach a $\approx 0.28\,\%$ precise scale setting, where the continuum limit extrapolation can be better controlled

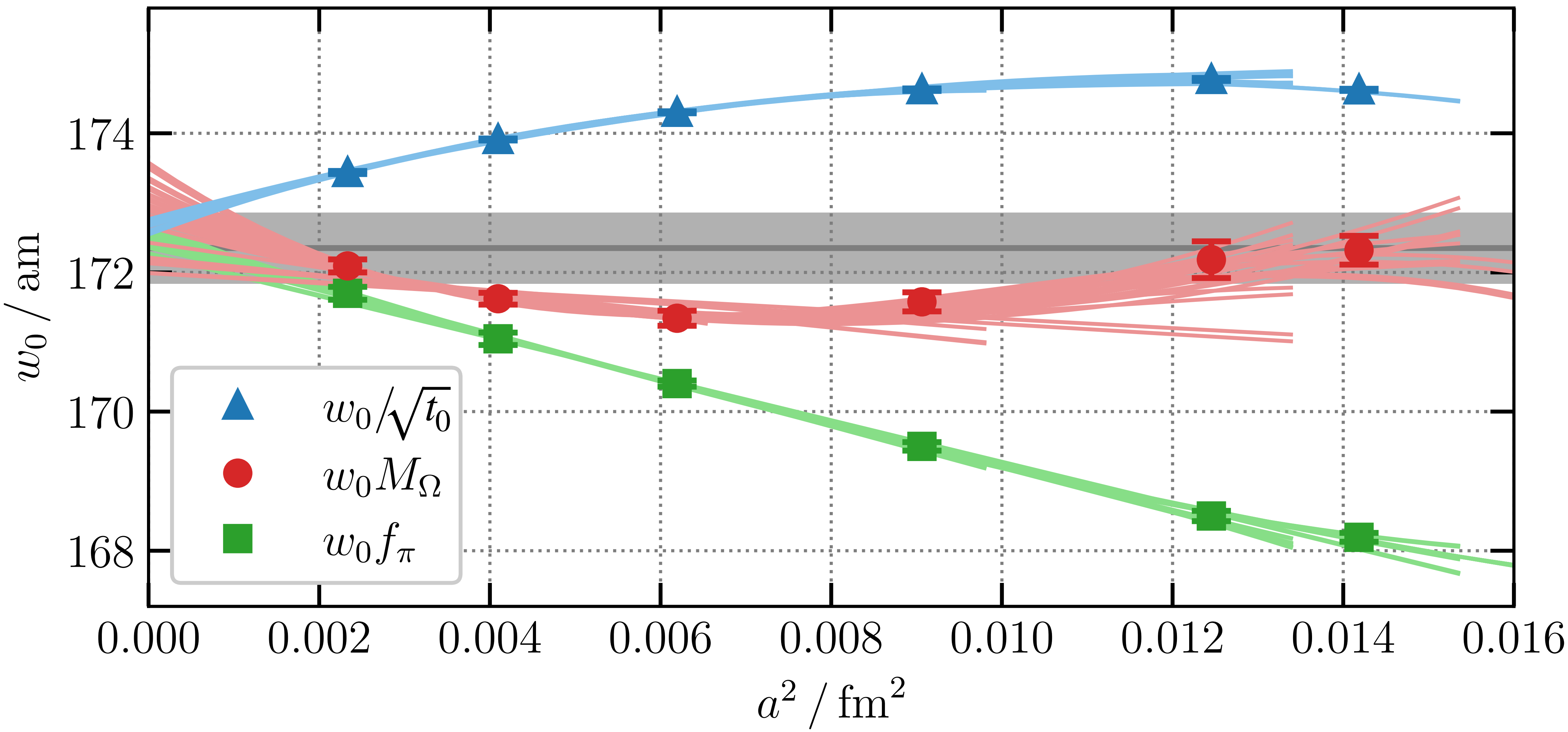
- The consistency of our w_0 determinations with these two physical inputs after unblinding is remarkable

- Further investigations on the impact of multi-hadron scattering states in the extraction of M_Ω are ongoing

- We are working on reducing the QCD error on w_0 as well as on improving the QED valence-valence contribution to $\mathcal{F}_\pi$

Backup slides

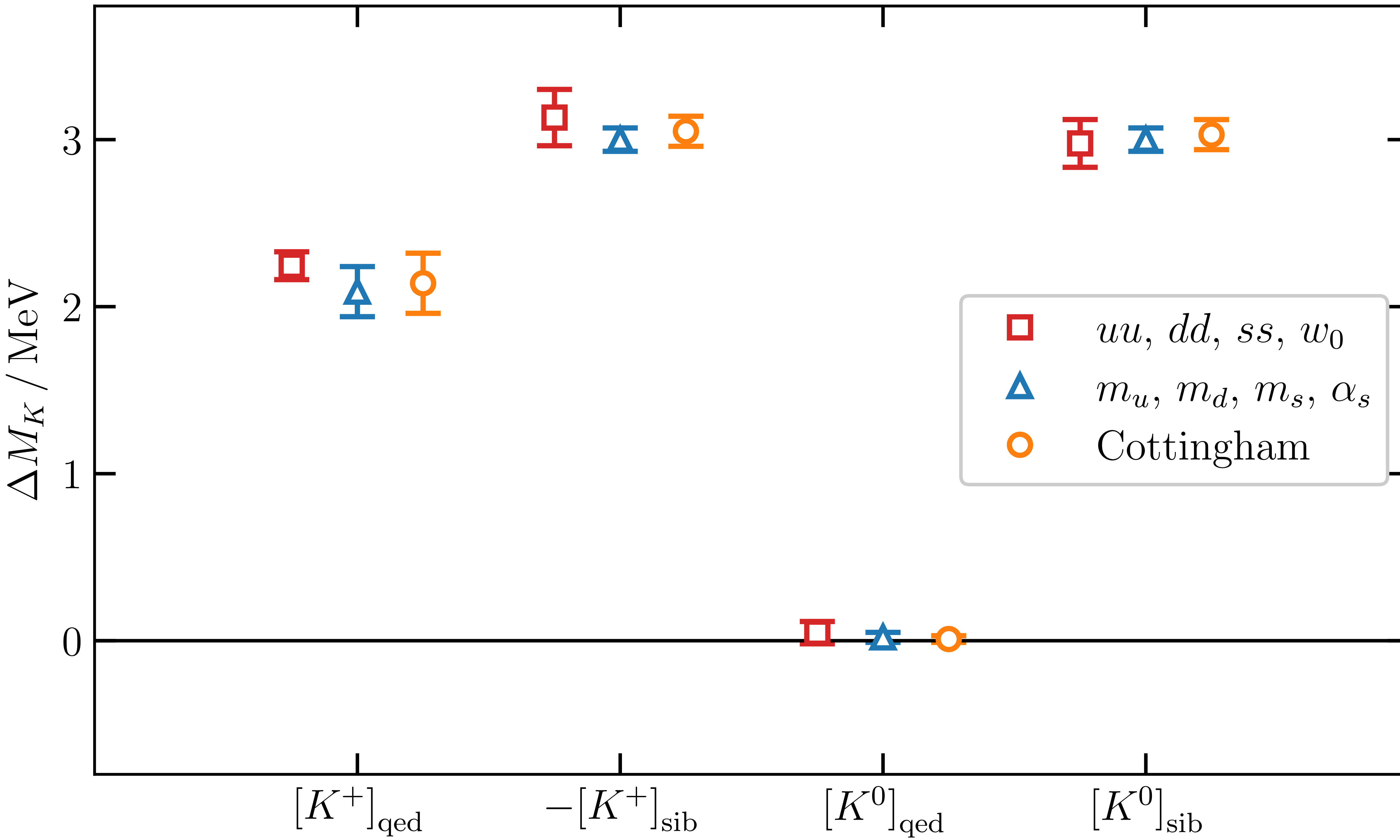
Comparison plot for scale setting quantities



$\sqrt{t_0} = 0.1416(^{+8}_{-5})$ fm $M_\Omega = 1672.45$ MeV $f_\pi = 130.5$ MeV

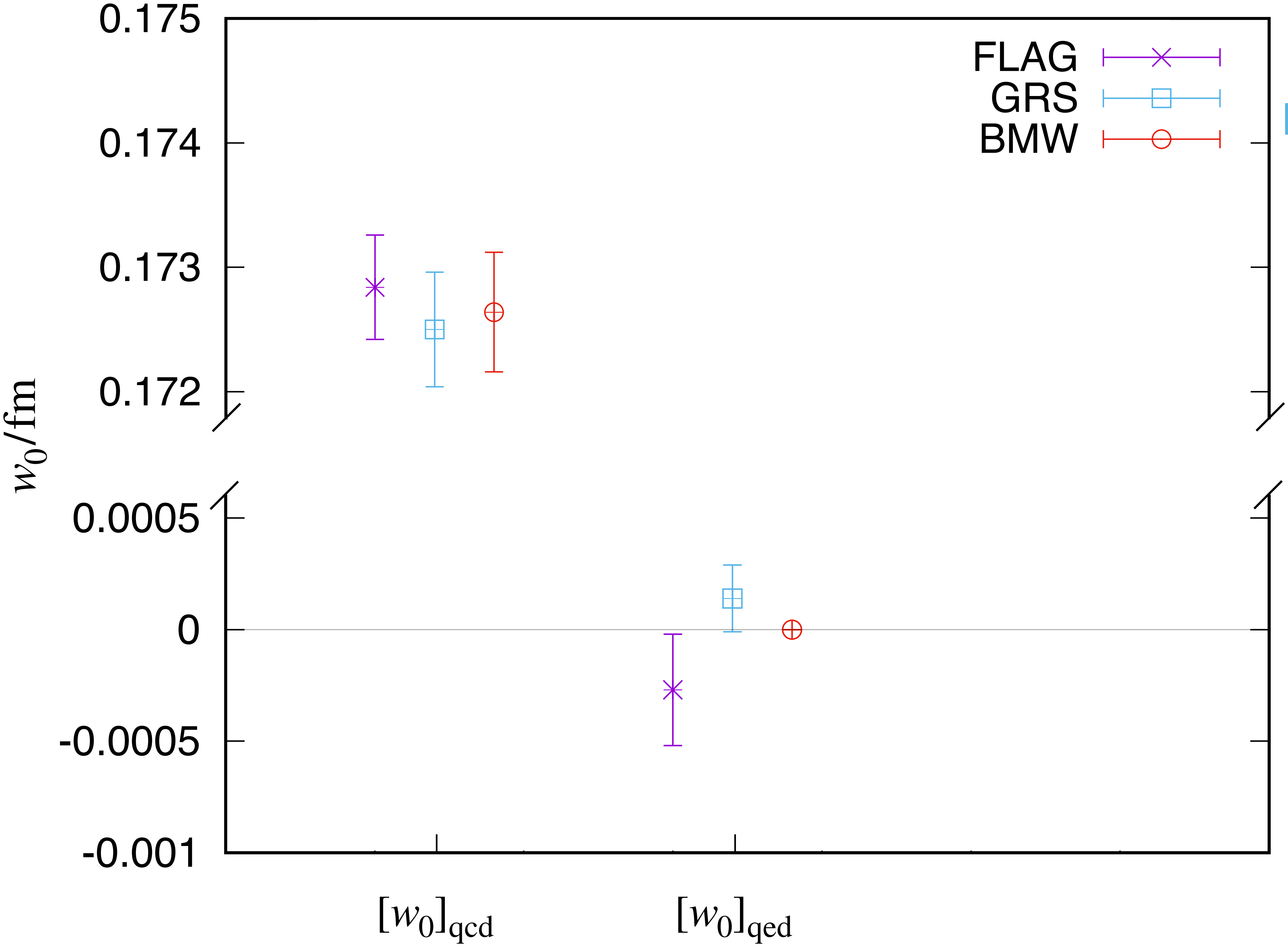
MILC '15 PDG FLAG

Kaon mass decomposition



BMW　　GRS [DG *et al.* '17]　　D. Stamen *et al.* '22

w_0 decomposition from w_0F_{ud}



GRS w/ $[M_{qq'}]_{\mathrm{qcd++}}$

Gauge ensembles

β	a [fm]	$L/a \times T/a$	tag	am_s	m_s/m_l	#confs
3.7000	0.1315	48×64	dir00	0.057291	27.899	904
3.7500	0.1191	56×96	dir00	0.049593	28.038	315
			dir01	0.049593	26.939	516
			dir02	0.051617	29.183	504
			dir03	0.051617	28.038	522
			dir05	0.055666	28.083	215
3.7753	0.1116	56×84	dir00	0.047615	27.843	510
			dir01	0.048567	28.400	505
			dir02	0.046186	26.469	507
			dir03	0.049520	27.852	385
3.8400	0.0952	64×96	dir00	0.043194	28.500	510
			dir02b	0.043194	30.205	436
			dir04	0.040750	28.007	1503
			dir05	0.039130	26.893	500
3.9200	0.0787	80×128	dir02	0.032440	27.679	506
			dir04	0.034240	27.502	512
			dir01b	0.032000	26.512	1001
			dir02b	0.032440	27.679	327
			dir03b	0.033286	27.738	1450
			dir04b	0.034240	27.502	500
4.0126	0.0640	96×144	phys1	0.026500	27.634	446
			phys2	0.026500	27.124	551
			phys1b	0.026500	27.634	2248
			phys2b	0.026500	27.124	1000
			phys3	0.027318	27.263	985
			phys4	0.027318	28.695	1750
4.1479	0.0483	128×192	phys1	0.019370	27.630	2792
			phys2	0.019951	27.104	2225