Update on the long-range radiative corrections to $\tau \to \pi\pi\nu_\tau$

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Colangelo, Cottini, MH, Holz PRL 136 (2026) 101903, JHEP 02 (2026) 181

[slight update compared to talk by S. Holz in Orsay]

Isospin-breaking corrections to $\tau \to \pi\pi\nu_\tau$: basics

Master formula for $\tau \to \pi\pi\nu_\tau(\gamma)$

$$\frac{1}{K_\Gamma(s)}\frac{d\Gamma}{ds}[\tau \to \pi\pi\nu_\tau(\gamma)] = \underbrace{S_{\text{EW}}^{\pi\pi}}_{\text{short distance}} \times \underbrace{\beta^3_{\pi\pi^0}}_{\text{phase space}} \times \underbrace{|f_+(s)|^2}_{\langle\pi\pi^0|j_W^\mu|0\rangle} \times \underbrace{G_{\text{EM}}(s)}_{\text{radiative corrections}}$$

- Short-distance corrections

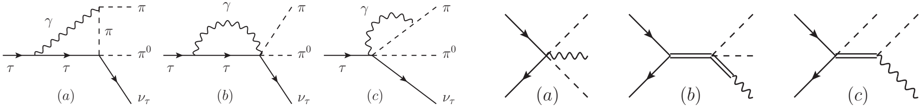
$$S_{\text{EW}}^{\pi\pi} = 1 + \frac{2\alpha}{\pi}\log\frac{M_Z}{m_\tau} + \cdots$$

- In isospin limit: $f_+(s)$ same as $F_\pi^V(s)$ (matrix element from $e^+e^- \to \pi^+\pi^-$)
- Radiative corrections subsumed into $G_{\text{EM}}(s)$ (similar to $\eta(s)$ in $e^+e^- \to \pi^+\pi^-$)

$$\sigma_{e^+e^-\to\pi^+\pi^-(\gamma)}(s) = \frac{1}{\mathcal{N}(s)\Gamma_e}\frac{d\Gamma_{\tau^\pm\to\pi^\pm\pi^0\nu_\tau(\gamma)}}{ds} \times \frac{\eta(s)}{S_{\text{EW}}^{\pi\pi}G_{\text{EM}}(s)}\frac{\big[\beta_{\pi\pi}(s)\big]^3}{\big[\beta_{\pi\pi^0}(s)\big]^3}\left|\frac{F_\pi^V(s)}{f_+(s)}\right|^2$$

- Procedure:
  1. Remove τ-specific IB corrections: $S_{\text{EW}}^{\pi\pi}G_{\text{EM}}(s)$ and phase space
  2. Apply corrections to matrix element to get from $f_+(s)$ to $F_\pi^V(s)$
  3. Add e^+e^- specific IB corrections ($\eta(s)$ and ρ–ω mixing)

Calculation in ChPT

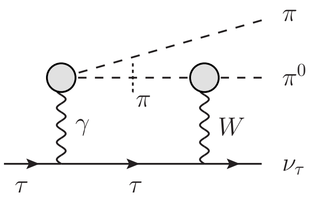


- Applies at low energies
- UV divergences removed by LECs

$$X_\ell \equiv \frac{4}{3}X_1 + X_6^r(\mu_\chi) - 4K_{12}^r(\mu_\chi)$$

$\hookrightarrow$ finite parts not predicted

- **Matching to lattice-QCD calculations** of $\langle \pi\pi^0 | j_{\text{em}}^\mu j_W^\nu | 0 \rangle$ Feng et al. 2020, Yoo et al. 2023
- Use $X_\ell(M_\rho) = 14 \times 10^{-3}$ Ma et al. 2021, improved matching following talk by V. Cirigliano
- Uncertainty in real emission dominated by F_A
- **Important update**: Local ChPT contribution dropped in previous work (including $\Delta X_\ell\big|_{\text{SD}} = -\frac{1}{4\pi^2}\log\frac{m_\tau^2}{M_\rho^2}$), **sizable numerical effect**



- Dominant correction from **pion pole** in $\langle \pi\pi^0 | j^\mu_{\text{em}} j^\nu_W | 0\rangle$

  $\hookrightarrow$ reduces to ChPT for point-like form factors

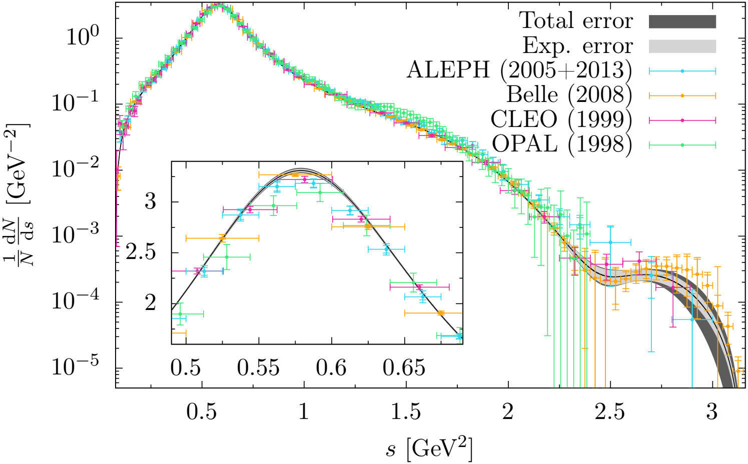
- **Matching at low energies**

$$f^{\text{full}}_{\text{loop}}(s,t) = f^{\text{disp}}_{\text{loop}}(s,t) - f^{\text{disp}}_{\text{loop}}(0,0) + f^{\text{ChPT}}_{\text{loop}}(0)$$

  $\hookrightarrow$ ensures that IR structure and chiral logs are correct

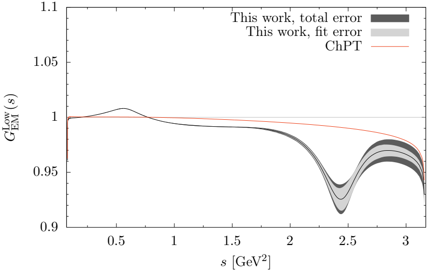
- Checked that limits match onto ChPT, including narrow-width limit and $M_\rho \to \infty$ for UV divergence

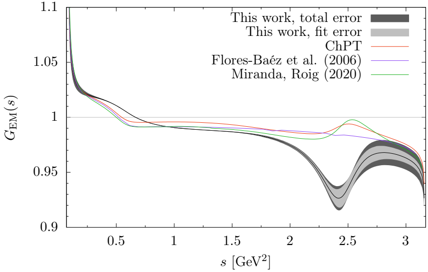
Fits to the τ spectrum



- Dispersive representation of $f_+(s)$, self-consistent iteration with $G_{\rm EM}(s)$, updates:
  - Investigated alternative implementations of ρ', ρ'' $\Rightarrow$ minor effects
  - Investigated fit quality: (mild) tension between threshold region and ρ peak

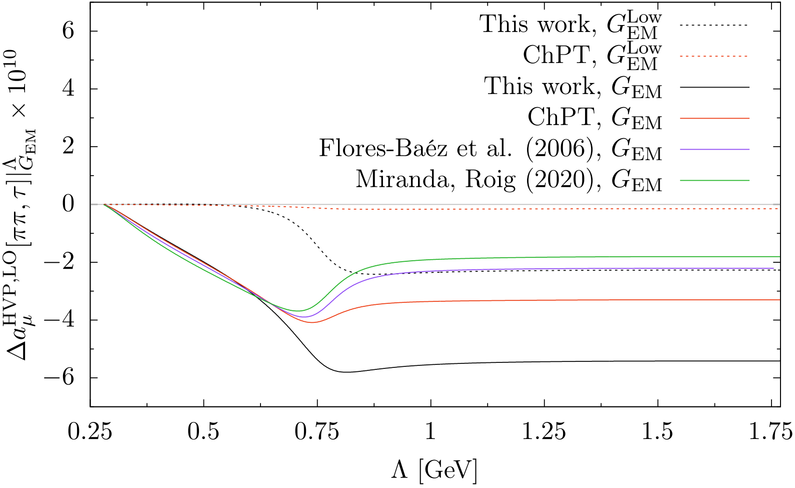
Results





	DHLMZ-23	LMR-24	WP25	This work
Phase space	-7.88	-7.52	$-7.7(2)$	$-7.74(5)$
$S_{\mathrm{EW}}^{\pi\pi}$	$-12.21(15)$	$-12.16(15)$	$-12.2(1.3)$	$-12.2(1.3)$
$G_{\mathrm{EM}}^{\mathrm{full}}$	$-1.92(90)$	$(-1.67)^{+0.60}_{-1.39}$	$-2.0(1.4)$	$-5.4(5)$
Sum	$-22.01(91)$	$(-21.35)^{+0.62}_{-1.40}$	$-21.9(1.9)$	$-25.3(1.4)$
Full	–	–	–	$-24.8(1.4)$

Results


This work, $G_{\text{EM}}^{\text{Low}}$
ChPT, $G_{\text{EM}}^{\text{Low}}$
This work, G_{EM}
ChPT, G_{EM}
Flores-Baéz et al. (2006), G_{EM}
Miranda, Roig (2020), G_{EM}
$\Delta a_\mu^{\text{HVP,LO}}[\pi\pi,\tau]|_{G_{\text{EM}}}^{\Lambda} \times 10^{10}$
Λ [GeV]
6
4
2
0
-2
-4
-6
0.25
0.5
0.75
1
1.25
1.5
1.75

Isospin breaking in the matrix element

- In ChPT/dispersion relations, $f_+(s)$ and $F_\pi^V(s)$ are well defined as the **current matrix elements**

$$\langle\pi^\pm(p')|j^\mu_{W\mp}(0)|\pi^0(p)\rangle = \mp\sqrt{2}(p+p')^\mu f_+(s) \qquad j^\mu_{W-} = \bar{u}\gamma^\mu d \qquad j^\mu_{W+} = \bar{d}\gamma^\mu u$$

$$\langle\pi^\pm(p')|j^\mu_{\text{em}}(0)|\pi^\pm(p)\rangle = \pm(p+p')^\mu F_\pi^V(s) \qquad j^\mu_{\text{em}} = \frac{1}{3}(2\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d - \bar{s}\gamma^\mu s)$$

with the long-distance radiative effects removed and the low-energy constants matched to LEFT

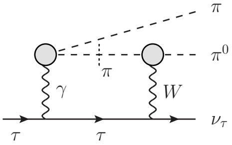
- Dispersive approach for remaining **IB in the matrix elements**
  1. IB in $\pi\pi$ scattering Cottini, Colangelo, Ruiz de Elvira 2025
  2. IB in $|f_+(s)/F_\pi^V(s)|$ by solving the (extended) unitarity relation work in progress

  $\hookrightarrow$ challenging program, likely limited by low-energy/subtraction constants

- Matching to lattice QCD could be very beneficial

  $\hookrightarrow$ need to figure out suitable scheme and matching procedure

Conclusions



- **Radiative corrections to $\tau \to \pi\pi\nu_\tau$**
  - Extended validity of $G_{\text{EM}}(s)$ beyond low-energy region
  - Structure-dependent virtual corrections are large
  - Important effect from local terms in ChPT
  - Partially canceled by increased integral over τ spectrum due to tension between threshold and ρ region

    $\hookrightarrow$ **new data on spectrum would be extremely valuable to clarify** BESIII, Belle II
- Next steps:
  - Improved matching to ChPT and short-distance corrections talk by V. Cirigliano
  - IB in $|f_+(s)/F_\pi^V(s)|$ using dispersion relations work in progress
  - IB in $|f_+(s)/F_\pi^V(s)|$ via matching to lattice QCD? talk by M. Bruno, discussion session