9th Plenary Workshop of the g-2 Theory Initiative
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Matching short- and long-distance corrections to $\tau \rightarrow \pi\pi\nu_\tau$ to NLL accuracy

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Motivation: rad. corr. to $\tau \to \pi\pi\nu_\tau$ and HVP

$$\boxed{\frac{d\Gamma_{\pi\pi[\gamma]}}{ds} = S_{\rm EW}^{\pi\pi}\, K_\Gamma(s)\, \beta^3_{\pi^\pm\pi^0}(s)\, \left|f_+(s)\right|^2\, G_{\rm EM}(s)}$$

See Martin Hoferichter's talk

$\boxed{\Delta a_\mu^{\text{HVP, LO}}[\pi\pi, \tau] \text{ (in units of } 10^{-10})}$

	DMZ	LMR	WP-2025	DISP
Phase space	-7.88	-7.52	$-7.7(2)$	$-7.74(5)$
$S_{\rm EW}^{\pi\pi}$	$-12.21(15)$	$-12.16(15)$	$-12.2(1.3)$	$-12.2(1.3)$
$G_{\rm EM}^{\rm full}$	$-1.92(90)$	$(-1.67)^{+0.60}_{-1.39}$	$-2.0(1.4)$	$-5.4(5)$
Sum	$-22.01(91)$	$(-21.35)^{+0.62}_{-1.40}$	$-21.9(1.9)$	$-25.3(1.4)$

DMZ 2312.02053

LMR 2411.07696

DISP 2510.26871 2511.07507

- WP-2025 assigned $\pm\alpha/\pi$ uncertainty to $S_{\rm EW}^{\pi\pi}$ due to uncontrolled scheme dependence [induced by γ_5 and evanescent operators] that should be canceled by corrections to the long-distance matrix element ($G_{\rm EM}$)
- After the dispersive improvement of $G_{\rm EM}$, this is the largest uncertainty in radiative corrections

Outline

- Semileptonic charged-current processes beyond leading logarithms
  - Generalities
  - Match 'LEFT' (= V-A Theory + QED + QCD) to χPT for pion processes
- Applications
  - Pion beta decay ($\pi^+ \to \pi^0 e^+ \nu_e$)
  - $\tau^- \to \pi^- \pi^0 \nu_\tau$ and implications for tau-based HVP

Based on VC, M. Hoferichter, N. Valori, Phys. Rev. Lett. 137 (2026) 5, 051902 [e-Print:2602.11253]

Semileptonic charged-current processes beyond leading logarithms

Semileptonic CC processes beyond leading logs

$$\mathcal{L}_{\rm Fermi} = -2\sqrt{2} G_F V_{ud}\, C_\beta^{(f)}(\mu, a)\, \underbrace{\bar{\ell}_L \gamma_\rho \nu_{\ell L}\, \bar{u}_L \gamma^\rho d_L}_{\equiv O_\beta} + \text{h.c.}$$

- Universal Wilson coefficient $C_\beta(\mu)$
  - Same for all quark and *lepton* flavors
  - Known to NLL in a class of perturbative schemes (MS-bar + NDA (γ_5) + evanescent op. choice) & RI-MOM

- Process-dependent matrix elements of O_β
  - $\tau \to \pi\pi\nu_\tau$, $\pi^+ \to \pi^0 e^+ \nu_e$, n & nuclear β decays
  - Needed to NLL accuracy [$O(\alpha\alpha_S)$], in the same scheme as Wilson Coefficient

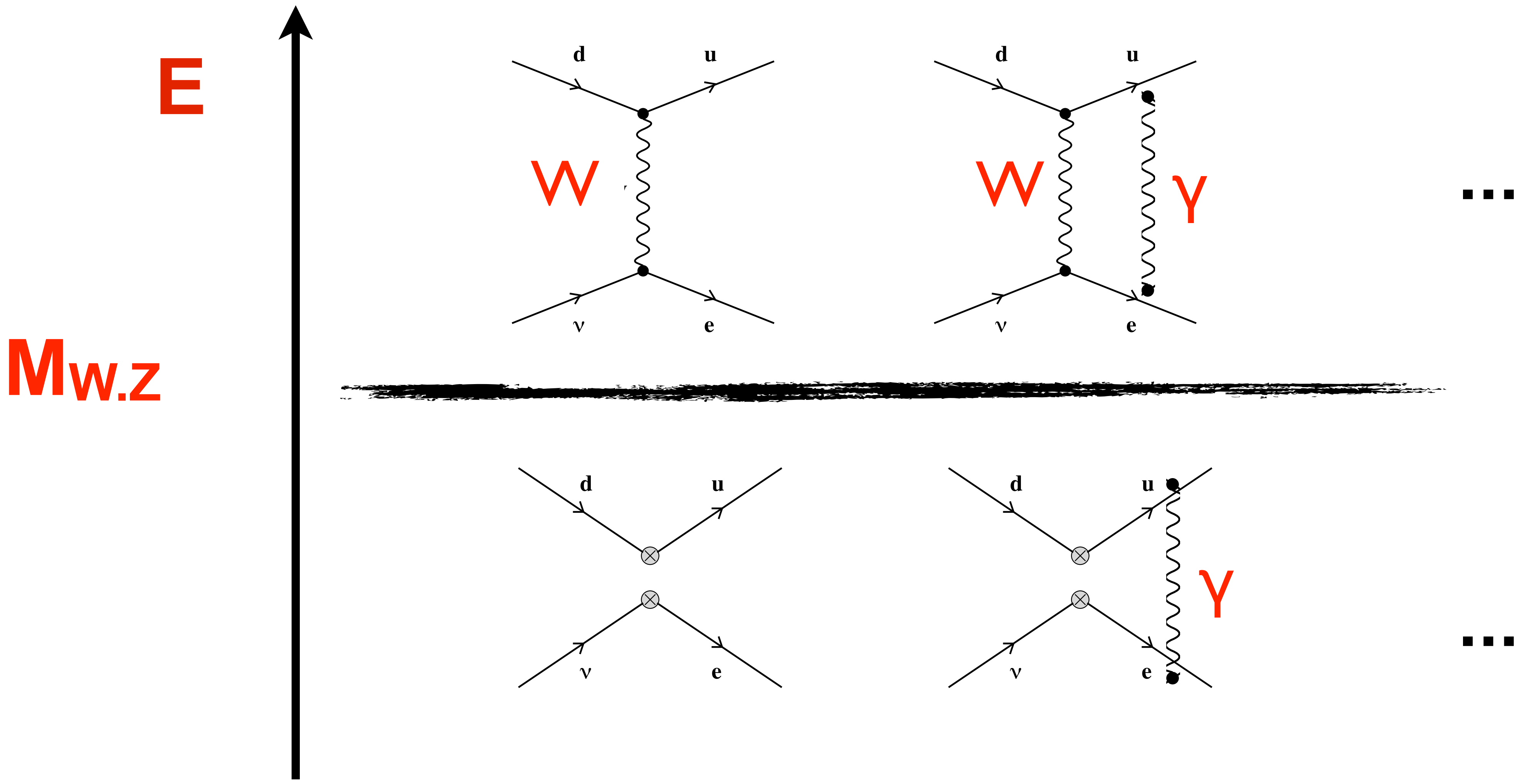
Only the product $C_\beta(\mu, a) \times \langle f|O_\beta|i\rangle(\mu, a)$ is physical (scale- and scheme-independent)

Wilson coefficient to NLL

$$\mathcal{L}_{\text{Fermi}} = -2\sqrt{2} G_F V_{ud}\, C_\beta^{(f)}(\mu, a)\, \bar{\ell}_L \gamma_\rho \nu_{\ell L}\, \bar{u}_L \gamma^\rho d_L + \text{h.c.}$$

$\mu \sim M_W$

$$C_\beta^{(5)}(\mu, a) = 1 + \frac{\alpha}{\pi} \log\frac{M_Z}{\mu} + \frac{\alpha}{4\pi} B(a) - \frac{\alpha\alpha_s}{4\pi^2} \log\frac{M_Z}{\mu} + \frac{\alpha\alpha_s}{(4\pi)^2} B_s(a) - \frac{\alpha\alpha_s}{4\pi^2}\frac{1}{s_W^2}\left(\frac{c_W^2}{s_W^2}\log\frac{M_W}{M_Z} + 1\right) + \mathcal{O}(\alpha^2, \alpha\alpha_s^2)$$



Evanescent scheme affects finite (non-log) terms and running

$$B(a) = \frac{a}{3} - 4 \qquad\qquad B_s(a) = \frac{41}{6} - \frac{14}{9} a$$

$$\gamma^\alpha \gamma^\rho \gamma^\beta P_L \otimes \gamma_\alpha \gamma_\rho \gamma_\beta P_L = 4(4 - a\epsilon)\gamma^\rho P_L \otimes \gamma_\rho P_L + E(a)$$

$$d = 4 - 2\epsilon$$

Wilson coefficient to NLL

$$\mathcal{L}_{\text{Fermi}} = -2\sqrt{2}G_F V_{ud}\, C_\beta^{(f)}(\mu, a)\, \bar{\ell}_L \gamma_\rho \nu_{\ell L}\, \bar{u}_L \gamma^\rho d_L + \text{h.c.}$$

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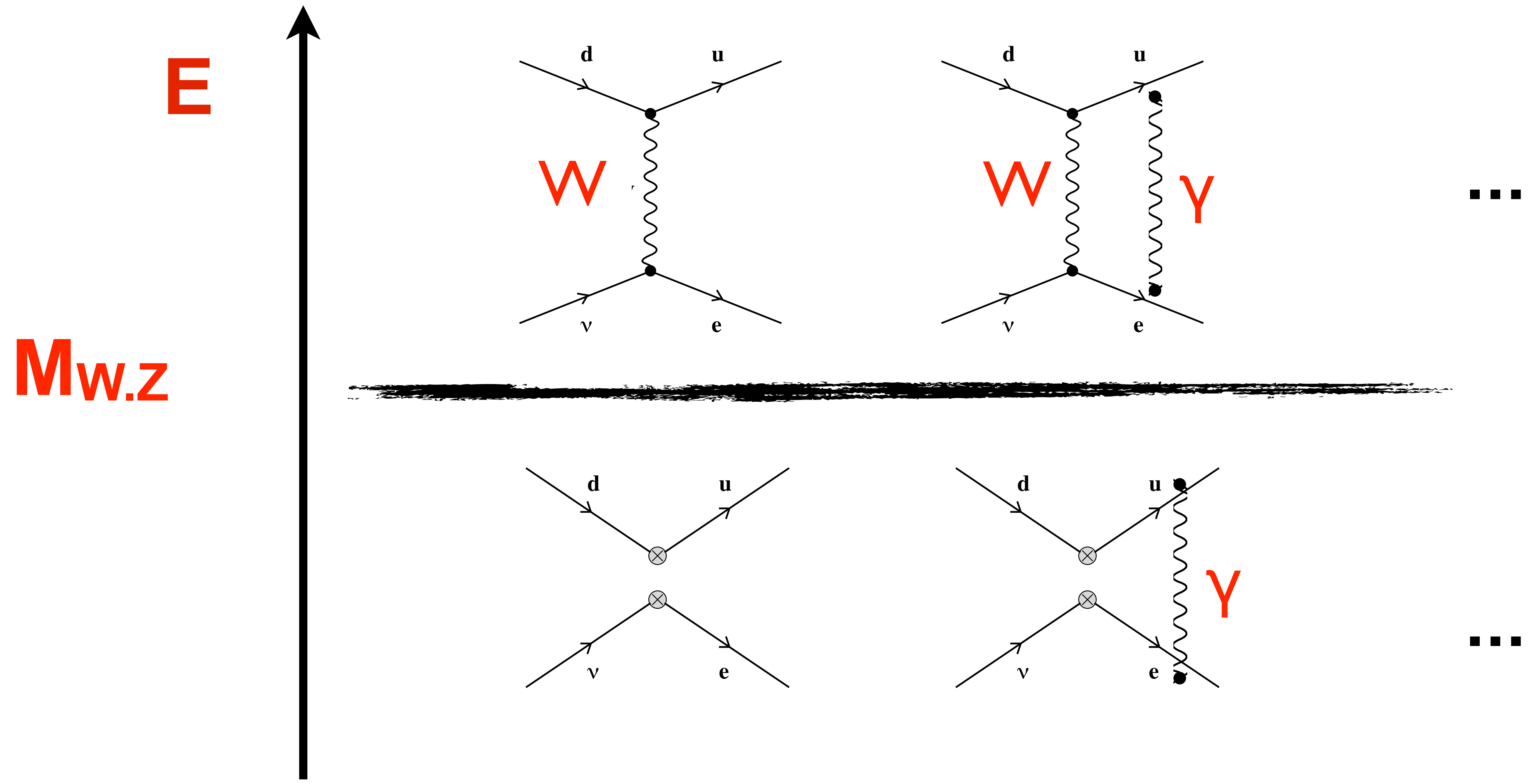
$$C_\beta^{(5)}(\mu, a) = 1 + \frac{\alpha}{\pi}\log\frac{M_Z}{\mu} + \frac{\alpha}{4\pi}B(a) - \frac{\alpha\alpha_s}{4\pi^2}\log\frac{M_Z}{\mu} + \frac{\alpha\alpha_s}{(4\pi)^2}B_s(a) - \frac{\alpha\alpha_s}{4\pi^2}\frac{1}{s_W^2}\left(\frac{c_W^2}{s_W^2}\log\frac{M_W}{M_Z} + 1\right) + \mathcal{O}(\alpha^2, \alpha\alpha_s^2)$$



Running: resummation of large logarithms

LL ~ $(\alpha \ln(M_Z/\mu))^n$ & $\alpha\,(\alpha_S \ln(M_Z/\mu))^n$ ← Sirlin 1982

NLL ~ $\alpha\,(\alpha \ln(M_Z/\mu))^n$ & $\alpha\,\alpha_S\,(\alpha_S \ln(M_Z/\mu))^n$

NLL: 2-loop anomalous dim. (α^2)
VC, W. Dekens, E. Mereghetti, O. Tomalak, 2306. 03138

NLLs: 3-loop anomalous dim. ($\alpha\alpha_S^2$)
M. Gorbahn, F. Moretti, S. Jaeger 2510.27648

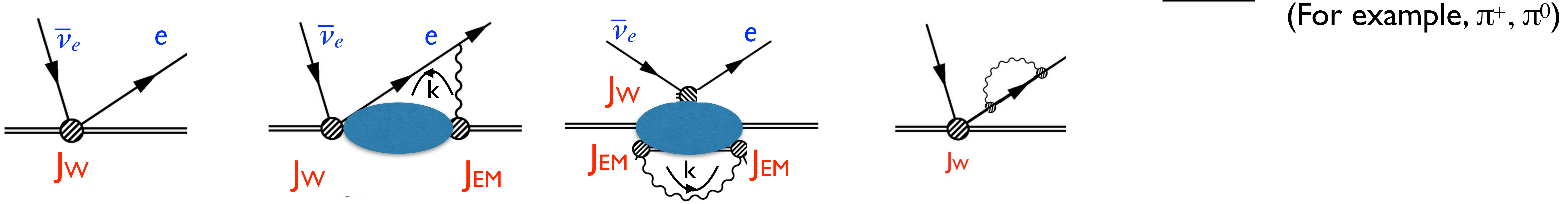
Conversion to RI-(S)MOM schemes

Gorbhan et al, 2209.05289 Boyle et al, 2606.04729

Process-dependent matrix element

$$\mathcal{L}_{\text{Fermi}} = -2\sqrt{2} G_F V_{ud}\, C_\beta^{(f)}(\mu, a)\, \bar{\ell}_L \gamma_\rho \nu_{\ell L}\, \bar{u}_L \gamma^\rho d_L + \text{h.c.}$$

$\langle f|$ $|i\rangle$ *Matrix elements to $O(\alpha)$*



- M.E. depends on hadronic structure & needs to reabsorb the scheme dependence of W.C. — a constant
- For $\tau \to \pi\pi\nu_\tau$, this can be achieved in the region of $\sqrt{s_{\pi\pi}} << \text{GeV}$, where χPT applies

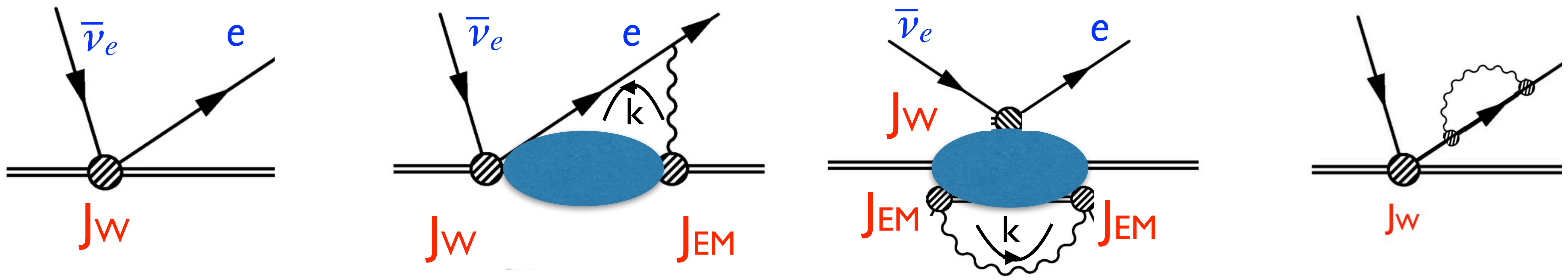
See talk by M. Hoferichter

$$f^{\text{full}}_{\text{loop}}(s,t) = f^{\text{disp}}_{\text{loop}}(s,t) - f^{\text{disp}}_{\text{loop}}(0,0) + f^{\text{ChPT}}_{\text{loop}}(0)$$

Process-dependent matrix element

$$\mathcal{L}_{\text{Fermi}} = -2\sqrt{2} G_F V_{ud}\, C_\beta^{(f)}(\mu, a)\, \boxed{\bar{\ell}_L \gamma_\rho \nu_{\ell L}\, \bar{u}_L \gamma^\rho d_L} + \text{h.c.}$$

$\langle f | \qquad | i \rangle$ *Matrix elements to O(α)*



hadron state
(For example, π^+, π^0)

- M.E. depends on hadronic structure & needs to reabsorb the scheme dependence of W.C. — a constant
- For $\tau \to \pi\pi\nu_\tau$, this can be achieved in the region of $\sqrt{s_{\pi\pi}}$ << GeV, where χPT applies

See talk by M. Hoferichter

$$f_{\text{loop}}^{\text{full}}(s,t) = f_{\text{loop}}^{\text{disp}}(s,t) - f_{\text{loop}}^{\text{disp}}(0,0) + \boxed{f_{\text{loop}}^{\text{ChPT}}(0)}$$

In this framework, structure-dependence *and* scheme-dependence are both captured by chiral LECs

χPT for semi-leptonic processes involving π's

$$\mathcal{L}_{\chi PT} = i\sqrt{2} G_F V_{ud}\, g_V^{\pi}(\mu_\chi)\, \bar{\nu}_{\ell L} \gamma_\rho \ell_L \left(\pi^0 \partial^\rho \pi^+ - \pi^+ \partial^\rho \pi^0\right) + \ldots$$

- g_V at $\mu_\chi \sim \Lambda_\chi$ encodes the effect of hard photons:

$$\boxed{g_V^{\pi}(\mu_\chi) \equiv C_\beta^{(3)}(\mu, a)\Big(1 - 2\pi\alpha_\chi(\mu_\chi) X_\ell(\mu_\chi, \mu, a)\Big)}$$

$$X_\ell(\mu_\chi, \mu, a) \equiv \left(X_6^r - 4K_{12}^r + \frac{4}{3}X_1^r\right)(\mu_\chi, \mu, a).$$

Combination of chiral LECs of $O(e^2p^2)$

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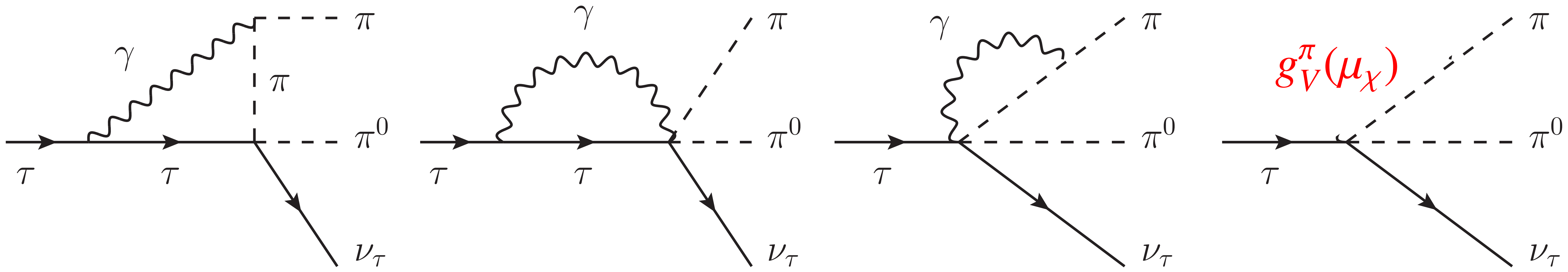
Combination of chiral LECs of $O(e^2p^2)$

- In physical amplitudes, the μ_χ scale cancels between loops and g_V



χPT for semi-leptonic processes involving π's

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Combination of chiral LECs of $O(e^2p^2)$

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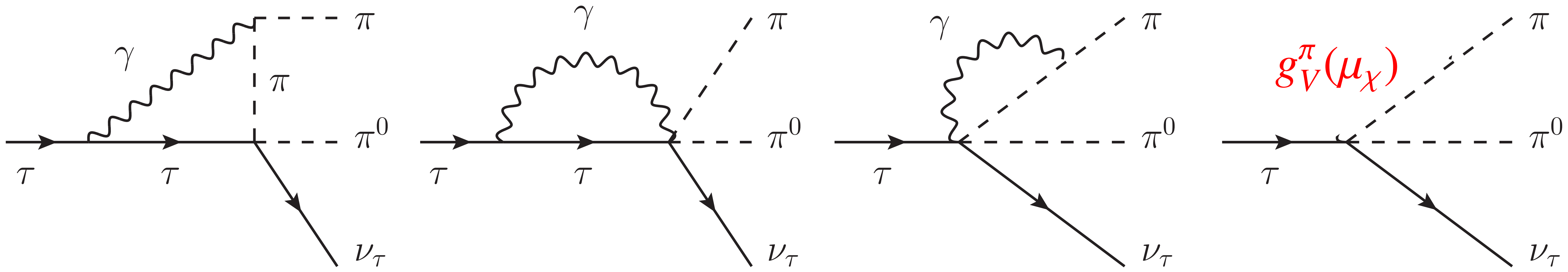
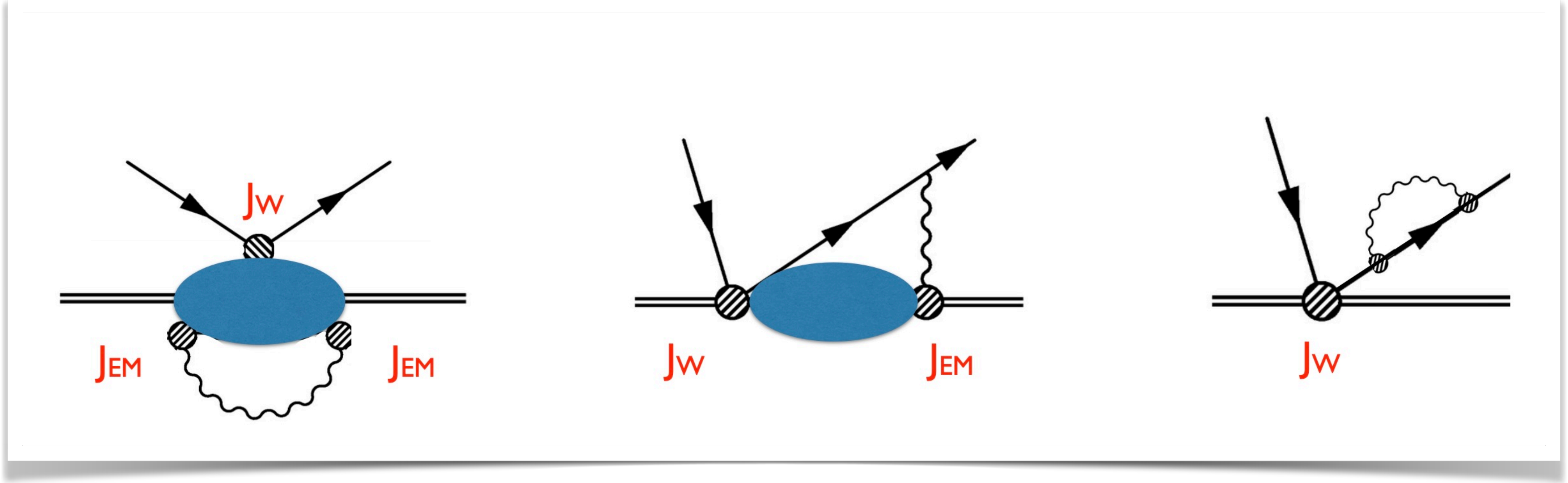


- Get g_V by matching Fermi theory to χPT, using lattice QCD input — simpler for $\pi^+ \to \pi^0 e^+ \nu_e$

Match 'LEFT' to ChPT

VC, M. Hoferichter, N. Valori 2602.11253

- Compute amplitude A($\pi^+ \to \pi^0 e^+ \nu_e$) ~ $C_\beta(\mu, a) \times \langle f|O_\beta|i\rangle(\mu, a)$ in 'LEFT' (= Fermi theory + QED + QCD) and ChPT



- Use dim-reg + MS-bar throughout
- Ward Identities → up to O(q_{ext}/m_π) the only non-perturbative input comes from '2pt functions'

Abers, Dicus, Norton, Quinn, Phys. Rev.167, 1462 (1968) → Sirlin 1978 RMP

$$W^{\mu\nu}(k) = i \int d^d x\, e^{-ik\cdot x} \langle h_f(p_f) \big| T\{W^\mu(0) J^\nu_{\rm em}(x)\} \big| h_i(p_i)\rangle \qquad W = V, A$$

Match 'LEFT' to ChPT

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- Compute amplitude A($\pi^+ \to \pi^0 e^+ \nu_e$) ~ $C_\beta(\mu, a) \times \langle f | O_\beta | i \rangle(\mu, a)$ in 'LEFT' (= Fermi theory + QED + QCD)

$$A_0 = -\sqrt{2} G_F V_{ud} C_\beta^{(3)} J^\mu_{\text{lept}} \langle h_f(p_f) | V_\mu(0) | h_i(p_i) \rangle$$

$$A = A_0 \left[\sqrt{Z_e} - 2e^2 i \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + i\epsilon)(\tilde{k}^2 - m_e^2 + i\epsilon)} \right]$$
$$+ 2i\sqrt{2} G_F V_{ud} C_\beta^{(3)} J^\mu_{\text{lept}} p_e^\nu e^2 \int \frac{d^d k}{(2\pi)^d} \frac{V_{\mu\nu}(k) + V^\alpha_\alpha(k)\, k_\mu k_\nu / k^2}{(k^2 + i\epsilon)(\tilde{k}^2 - m_e^2 + i\epsilon)}$$
$$+ \sqrt{2} G_F V_{ud} C_\beta^{(3)} \bar{u}(p_e) \frac{\gamma^\nu \gamma^\alpha \gamma^\mu - \gamma^\mu \gamma^\alpha \gamma^\nu}{2} P_L v(p_\nu) \times e^2 \int \frac{d^d k}{(2\pi)^d} \frac{k_\alpha A_{\mu\nu}(k)}{(k^2 + i\epsilon)^2 (\tilde{k}^2 - m_e^2 + i\epsilon)}$$

$$\tilde{k} = k + p_e$$

Match 'LEFT' to ChPT

VC, M. Hoferichter, N. Valori 2602.11253

- Compute amplitude A($\pi^+ \to \pi^0 e^+ \nu_e$) ~ $C_\beta(\mu, a) \times \langle f|O_\beta|i\rangle(\mu, a)$ in 'LEFT' (= Fermi theory + QED + QCD)

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$$+ 2i\sqrt{2} G_F V_{ud} C_\beta^{(3)} J^\mu_{\rm lept} p_e^\nu e^2 \int \frac{d^d k}{(2\pi)^d} \frac{V_{\mu\nu}(k) + V^\alpha_\alpha(k) k_\mu k_\nu / k^2}{(k^2 + i\epsilon)(\tilde{k}^2 - m_e^2 + i\epsilon)}$$

$$+ \sqrt{2} G_F V_{ud} C_\beta^{(3)} \bar{u}(p_e) \frac{\gamma^\nu \gamma^\alpha \gamma^\mu - \gamma^\mu \gamma^\alpha \gamma^\nu}{2} P_L v(p_\nu) \times e^2 \int \frac{d^d k}{(2\pi)^d} \frac{k_\alpha A_{\mu\nu}(k)}{(k^2 + i\epsilon)^2 (\tilde{k}^2 - m_e^2 + i\epsilon)}$$

<VV>
2pt function

$\tilde{k} = k + p_e$

Abers, Dicus, Norton, Quinn, PR 167, 1462 (1968)

- Leading term (unsuppressed by $q_{\rm ext}/m_\pi$) given by IR-singular Born terms $V^{\rm IR}_{\mu\nu}(k) = \langle h_f(p_f)|V_\mu(0)|h_i(p_i)\rangle \dfrac{v_\nu}{v \cdot k} + O(k^0)$
- Same in LEFT and χPT! Contributes ~ # $\log(\mu/\mu_\chi)$ to matching

Match 'LEFT' to ChPT

VC, M. Hoferichter, N. Valori 2602.11253

- Compute amplitude A($\pi^+ \to \pi^0 e^+ \nu_e$) ~ $C_\beta(\mu, a) \times \langle f|O_\beta|i\rangle(\mu, a)$ in 'LEFT' (= Fermi theory + QED + QCD)

$$A_0 = -\sqrt{2} G_F V_{ud} C_\beta^{(3)} J^\mu_{\text{lept}} \langle h_f(p_f)|V_\mu(0)|h_i(p_i)\rangle$$

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$$+ 2i\sqrt{2} G_F V_{ud} C_\beta^{(3)} J^\mu_{\text{lept}} p_e^\nu e^2 \int \frac{d^d k}{(2\pi)^d} \frac{V_{\mu\nu}(k) + V^\alpha_\alpha(k)\, k_\mu k_\nu / k^2}{(k^2 + i\epsilon)(\tilde{k}^2 - m_e^2 + i\epsilon)}$$

$$+ \sqrt{2} G_F V_{ud} C_\beta^{(3)} \bar{u}(p_e) \frac{\gamma^\nu \gamma^\alpha \gamma^\mu - \gamma^\mu \gamma^\alpha \gamma^\nu}{2} P_L v(p_\nu) \times e^2 \int \frac{d^d k}{(2\pi)^d} \frac{k_\alpha A_{\mu\nu}(k)}{(k^2 + i\epsilon)^2 (\tilde{k}^2 - m_e^2 + i\epsilon)}$$

<AV>
2pt function

$\tilde{k} = k + p_e$

- Nonperturbative input not dominated by IR (famous γW box) + dependence on evanescent scheme

Match 'LEFT' to ChPT

VC, M. Hoferichter, N. Valori 2602.11253

- Compute amplitude A($\pi^+ \to \pi^0 e^+ \nu_e$) ~ $C_\beta(\mu, a) \times \langle f | O_\beta | i \rangle (\mu, a)$ in 'LEFT' (= Fermi theory + QED + QCD)

$$A_0 = -\sqrt{2} G_F V_{ud} C_\beta^{(3)} J^\mu_{\rm lept} \langle h_f(p_f) | V_\mu(0) | h_i(p_i) \rangle$$

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$$+ \sqrt{2} G_F V_{ud} C_\beta^{(3)} \epsilon^{\nu\alpha\mu\rho} J_\rho^{\rm lept} e^2 \int \frac{d^4 k}{(2\pi)^4} \frac{k_\alpha \left[A_{\mu\nu}(k) - A^{\rm OPE}_{\mu\nu}(k, \mu_0) \right]}{(k^2 + i\epsilon)^2 (\tilde{k}^2 - m_e^2 + i\epsilon)}$$ + finite scheme-dependent

<AV>
2pt function

$$\tilde{k} = k + p_e$$

- Nonperturbative input not dominated by IR (famous γW box) + dependence on evanescent scheme
- Trick (= new renormalization scheme): add and subtract UV component of integrand, dictated by $A^{\rm OPE}_{\mu\nu}$
- Integrate subtracted term in d=4 and OPE term in d≠4 → cancel scheme dependence of Wilson Coefficient

Match 'LEFT' to ChPT

VC, M. Hoferichter, N. Valori 2602.11253

- Compute amplitude A($\pi^+ \to \pi^0 e^+ \nu_e$) ~ $C_\beta(\mu, a) \times \langle f | O_\beta | i \rangle (\mu, a)$ in 'LEFT' (= Fermi theory + QED + QCD)

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<AV>
2pt function

$$\tilde{k} = k + p_e$$

- Nonperturbative input not dominated by IR (famous γW box) + dependence on evanescent scheme
- Trick (=
- Integrat ... fficient

$A_{\mu\nu}$ absent in χPT loops to $O(e^2 p^2)$: local (LEC) contribution to the matching condition

Result for vector coupling

VC, M. Hoferichter, N. Valori 2602.11253

$$g_V^\pi(\mu_\chi) = \left[1 + \bar{\Box}_\pi^{V(3)}(\mu_0)\right] \times \left[1 + \frac{3\alpha^{(3)}(\mu)}{8\pi}\left(1 - \log\frac{\mu_\chi^2}{\mu^2}\right)\right] \times \left[Z^{s.c.}_{O,\dot{O}}(\mu, \mu_0, a)\right]^{-1} \times C_\beta^{(3)}(\mu, a)$$

<AV> 2pt function (γW box)

<VV> 2pt function

Convert MS-bar to OPE-subtracted scheme

W.C. in MS-bar

M. Gorbahn, F. Moretti, S. Jaeger 2510.27648

VC, W. Dekens, E. Mereghetti, O. Tomalak, 2306. 03138

$$\bar{\Box}_\pi^V(\mu_0) = \frac{3\alpha}{2\pi}\int_0^\infty \frac{dQ^2}{Q^2}\left[M_\pi(Q^2) - \frac{1}{12}\frac{Q^2}{Q^2+\mu_0^2}\left(1 - \frac{\alpha_s(\mu_0)}{\pi}\right)\right]$$

Related to T_3 appearing in

$A_{\mu\nu}(q) \supset i\epsilon_{\mu\nu\alpha\beta}q^\alpha p_\pi^\beta T_3(q_0, q^2)$

& calculated in LQCD

Result for vector coupling

VC, M. Hoferichter, N. Valori 2602.11253

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Convert MS-bar to OPE-subtracted scheme

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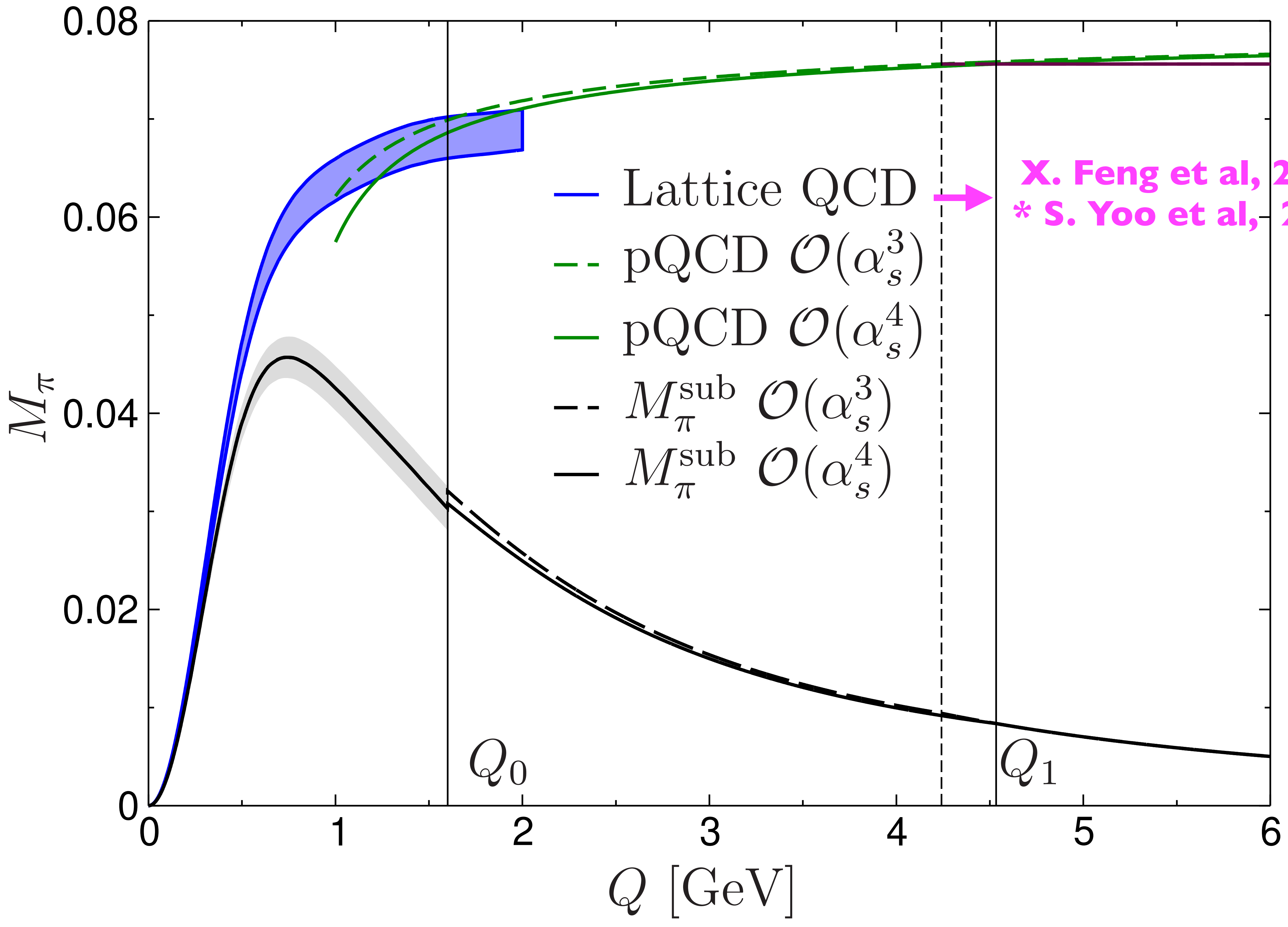

X. Feng et al, 2003.09798
* S. Yoo et al, 2305.03198

Result for vector coupling

VC, M. Hoferichter, N. Valori 2602.11253

$$g_V^\pi(\mu_\chi) = \left[1 + \bar{\Box}_\pi^{V(3)}(\mu_0)\right] \times \left[1 + \frac{3\alpha^{(3)}(\mu)}{8\pi}\left(1 - \log\frac{\mu_\chi^2}{\mu^2}\right)\right] \times \left[Z^{s.c.}_{O,O}(\mu,\mu_0,a)\right]^{-1} \times C_\beta^{(3)}(\mu,a)$$

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Convert MS-bar to OPE-subtracted scheme

W.C. in MS-bar

$$\bar{\Box}_\pi^V(\mu_0) = \frac{3\alpha}{2\pi}\int_0^\infty \frac{dQ^2}{Q^2}\left[M_\pi(Q^2) - \frac{1}{12}\frac{Q^2}{Q^2+\mu_0^2}\left(1 - \frac{\alpha_s(\mu_0)}{\pi}\right)\right]$$

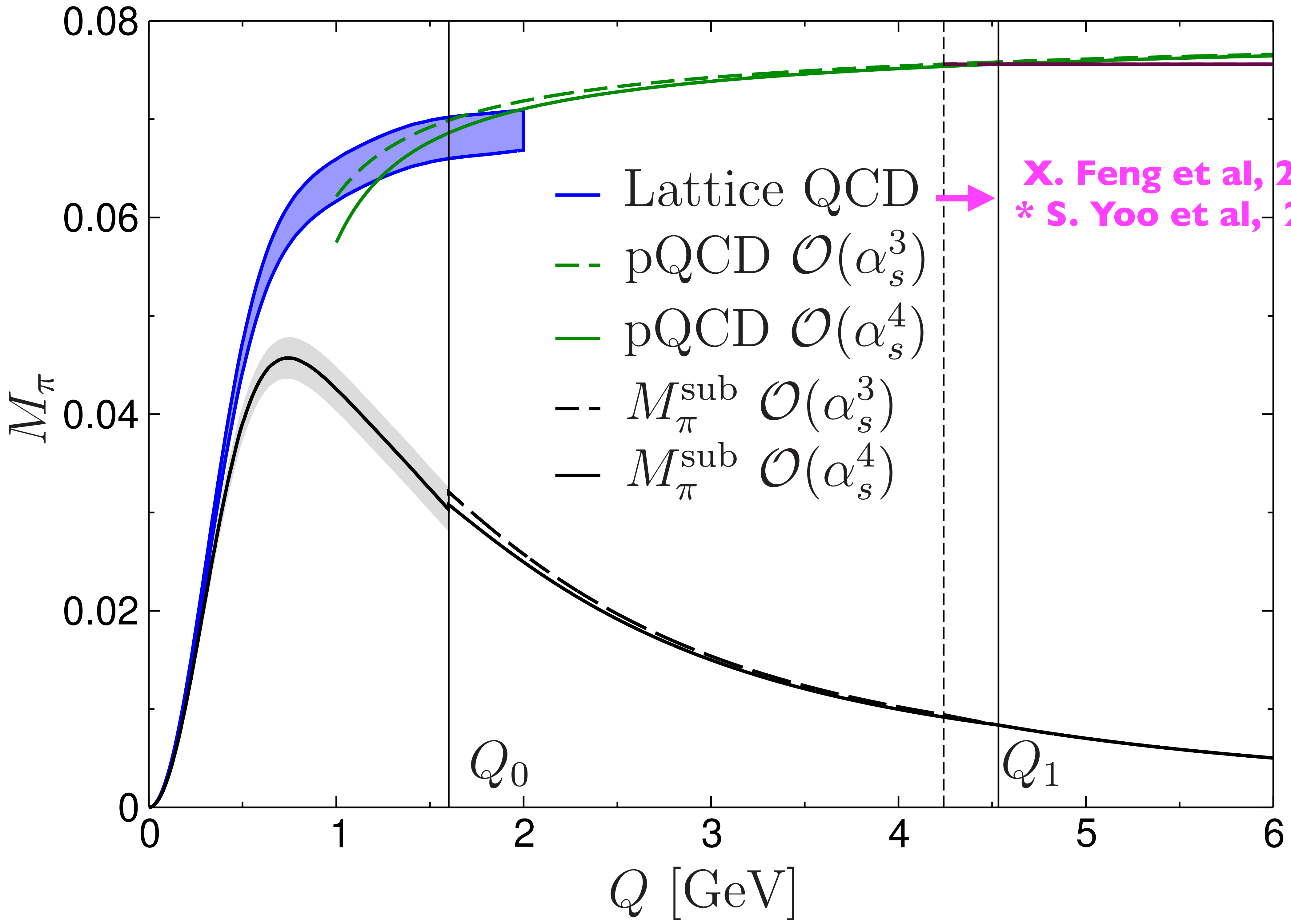
Related to T_3 appearing in

$A_{\mu\nu}(q) \supset i\epsilon_{\mu\nu\alpha\beta}q^\alpha p_\pi^\beta T_3(q_0, q^2)$

& calculated in LQCD

$$\boxed{g_V^\pi(M_\rho) = 1.01084(2)_{\bar{\Box}_\pi}(3)_\mu[4]_{\text{tot}}}$$

Average input of both LQCD calculations



X. Feng et al, 2003.09798
* S. Yoo et al, 2305.03198

Applications:
pion beta decay & $\tau \to \pi\pi\nu_\tau$

Pion β decay

VC, M. Hoferichter, N. Valori 2602.11253

$$\Gamma[\pi^+ \to \pi^0 e^+ \nu_e(\gamma)] = \frac{G_F^2 |V_{ud}|^2 M_\pi^5 |f_+^\pi(0)|^2}{64\pi^3}(1 + \Delta_{\mathrm{RC}}^{\pi\ell}) I_{\pi\ell}$$

- Form factor & phase space : $f_+^\pi(0) \simeq 1 - 7 \times 10^{-6}$ $\quad I_{\pi\ell} = 7.3767(41) \times 10^{-8}$

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$$\Delta_{\mathrm{RC}}^{\pi\ell} = \left(g_V^\pi(\mu_\chi)\right)^2 \left(1 + \frac{3\alpha_\chi(\mu_\chi)}{2\pi}\log\frac{\mu_\chi}{2E_0} + \Delta I_{\pi\ell}\right) - 1$$
$$= 0.03403(4)_{\bar{\Box}_\pi}(5)_\mu(5)_{\mu_\chi}(8)_{\mathrm{ChPT}}[11]_{\mathrm{tot}}$$

μ: varied around all thresholds
2510.27648

μ_χ: varied around M_ρ and $2E_0$

ChPT: $\frac{\alpha}{\pi}\left(\frac{M_\pi}{M_\rho}\right)^2$

Difference from $0.0332(1)_{\mathrm{box}}(3)_{\mathrm{HO, QED}}$ [2003.09798] understood in terms of (N)LL and g_V running in ChPT

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- Extraction of V_{ud}:

$$V_{ud}^\pi = 0.97346(281)_{\rm Br}(9)_{\tau_\pi}(5)_{\Delta_{\rm RC}^{\pi\ell}}(27)_{I_{\pi\ell}}[283]_{\rm tot}$$

Reduced rad. corr. uncertainty by factor of 3, below all experimental uncertainties (BR, pion masses, lifetime)

Radiative corrections to $\tau \to \pi\pi\nu_\tau$

VC, M. Hoferichter, N. Valori 2602.11253

$$\frac{d\Gamma_{\pi\pi[\gamma]}}{ds} = S_{\mathrm{EW}}^{\pi\pi}\, K_\Gamma(s)\, \beta^3_{\pi^\pm\pi^0}(s) \left|f_+(s)\right|^2 G_{\mathrm{EM}}(s)$$

$$K_\Gamma(s) = \frac{\Gamma_e |V_{ud}|^2}{2m_\tau^2}\left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + \frac{2s}{m_\tau^2}\right)$$

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$$\boxed{S_{\rm EW}^{\pi\pi} = S_{\rm EW}(m_\tau) - \Delta S^e_{\rm EW} \equiv 1 + \Delta\bar{S}^{\pi\pi}_{\rm EW}}$$

$$S_{\rm EW}(\mu) = \left[C_\beta^{\rm LL}(\mu)\right]^2 \qquad \Delta S^e_{\rm EW} = \frac{\alpha}{2\pi}\left(\frac{25}{4} - \pi^2\right)$$

$$\boxed{G_{\rm EM}(s) = G_0(s) + \Delta\bar{G}}$$

$$\Delta\bar{G} = -4\pi\alpha\left(\bar{X}_\ell(m_\rho) - \frac{1}{4\pi^2}\log\frac{m_\tau^2}{m_\rho^2}\right) \qquad \bar{X}_\ell(m_\rho) = 14 \times 10^{-3}$$

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$S_{\rm EW}(\mu) = \left[C_\beta^{\rm LL}(\mu)\right]^2$ $\Delta S^e_{\rm EW} = \frac{\alpha}{2\pi}\left(\frac{25}{4}-\pi^2\right)$	$\left[g_V^\pi(\mu_\chi)\right]^2 = \left[C_\beta^{\rm NLLs}(\mu,a)\right]^2\left[1-2\pi\alpha X_\ell(\mu,\mu_\chi,a)\right]^2$
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$\Delta\bar{G} = -4\pi\alpha\left(\bar{X}_\ell(m_\rho) - \frac{1}{4\pi^2}\log\frac{m_\tau^2}{m_\rho^2}\right) \qquad \bar{X}_\ell(m_\rho) = 14\times 10^{-3}$	$\Delta\left(S^{\pi\pi}_{\rm EW} G_{\rm EM}(s)\right)_{\text{NEW- OLD}} = \Delta S^{\pi\pi}_{\rm EW} - (\Delta\bar{S}^{\pi\pi}_{\rm EW} + \Delta\bar{G})$

Impact on HVP

VC, M. Hoferichter, N. Valori 2602.11253

- Impact of a constant shift in $S_{\rm EW}^{\pi\pi}G_{\rm EM}(s)$ can be evaluated by scaling the 'old' correction induced by $S_{\rm EW}^{\pi\pi}$

$$\Delta a_\mu[\pi\pi,\tau] - \Delta\bar{a}_\mu[\pi\pi,\tau] = \left(\frac{\Delta S_{\rm EW}^{\pi\pi} - \Delta\bar{G}_{\rm EM}}{\Delta\bar{S}_{\rm EW}^{\pi\pi}} - 1\right)\Delta\bar{a}_\mu[S_{\rm EW}^{\pi\pi}] = -0.07(4)\times 10^{-10}$$

$$\Delta\bar{a}_\mu[S_{\rm EW}^{\pi\pi}] = -12.166(56)(8)\times 10^{-10}$$

- Putting together tau-specific corrections [phase space and $S_{\rm EW}^{\pi\pi}G_{\rm EM}(s)$]

Colangelo, Cottini, Hoferichter, Holz, 2510.26871, 2511.07507

$$\Delta a_\mu[\pi\pi,\tau] = -24.9(1)_{\rm exp}(5)_{\rm th}(1)_{\rm SD} \times 10^{-10}$$

- Uncertainty from short-distance matching reduced from 1.3×10^{-10} to 0.1×10^{-10}
- Dominant uncertainty arises from the ratio of form factors, and was estimated as ~ 5×10^{-10} in WP-2025

Challenge: isolating ‘form factor’

$$\frac{d\Gamma_{\pi\pi[\gamma]}}{ds} = S^{\pi\pi}_{\rm EW} K_\Gamma(s)\, \beta^3_{\pi^\pm\pi^0}(s) \left|f_+(s)\right|^2 G_{\rm EM}(s)$$

- Factorization into Wilson coefficient × hadronic threshold correction and Hadronic Matrix Element is now under control at NLL

- Further factorization into ‘long-distance radiative corrections’ and ‘form factor’ (matrix element of the hadronic current, which still contains IB) entails some additional choice (scheme)
  - ChPT and dispersive approach define form factor corrections as whatever is left after removing from the amplitude the long-distance radiative corrections and short-distance effects (LECs)
  - To compare, need a definition that can be implemented in any method, including Lattice QCD

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Conclusions

- Described semileptonic CC processes involving low-energy pions beyond leading logarithms
  - Matched 'LEFT' to χPT to NLLs ($\alpha\alpha_S$) accuracy
  - Used lattice QCD results to evaluate the non-perturbative input

- Pion beta decay ($\pi^+ \to \pi^0 e^+ \nu_e$):
  - Reduced theory uncertainty by factor of 3, now below the smallest expt. uncertainties
  - Even 'cleaner' avenue to extract V_{ud} — need ten-fold exp. improvement (PIONEER)

- $\tau \to \pi\pi\nu_\tau$ and implications for tau-based HVP:
  - Uncertainty from short-distance matching reduced from 1.3×10^{-10} to 0.1×10^{-10}
  - Dominant uncertainty is in the form factor ratio: need definition of 'form factor' corrections that can be implemented in multiple methods, including LQCD

Backup

Impact of NLL corrections

- From weak scale to hadronic scale

$$\Delta g_V\big|_{LL(\alpha)} \sim +3 \cdot 10^{-4}$$
$$\Delta g_V\big|_{LL(\alpha\alpha_s)} \sim -1.6 \cdot 10^{-4}$$
$$\Delta g_V\big|_{NLL(\alpha)} \sim -1.0 \cdot 10^{-4}$$
$$\Delta g_V\big|_{NLL(\alpha\alpha_s)} \sim -0.6 \cdot 10^{-4}$$

- From hadronic scale to endpoint

$$\Delta g_V\big|_{LL(\alpha)} \sim +1.7 \cdot 10^{-4}$$
$$\Delta g_V\big|_{NLL(\alpha)} \sim +0.5 \cdot 10^{-4}$$

More details on matching

$$g_V^{\pi}(\mu_\chi) = \left[1 + \bar{\Box}_\pi^{V(3)}(\mu_0)\right] \times \left(Z_{O,O}^{s.c.}(\mu, \mu_0, a)\right)^{-1}$$
$$\times \left[1 + \frac{3\alpha^{(3)}(\mu)}{8\pi}\left(1 - \log\frac{\mu_\chi^2}{\mu^2}\right)\right] \times C_\beta^{(3)}(\mu, a).$$

$$Z_{O,O}^{s.c.}(\mu, \mu_0, a) = 1 + \frac{\alpha}{4\pi}\left[\frac{a}{3} - \frac{5}{4} - \log\frac{\mu}{\mu_0}\right.$$
$$\left. + \frac{\alpha_s}{4\pi}\left(-\frac{14a}{9} + \frac{29}{6} + 4\log\frac{\mu}{\mu_0}\right)\right].$$