Isospin-breaking effects in inclusive hadronic τ data for the muon $(g-2)$

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2607.00564, 2606.04729
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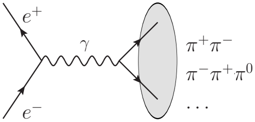
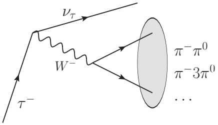
τ DECAYS FOR THE MUON $g-2$



Isospin breaking corrections
connect τ with HVP
for muon $g-2$
[Alemani et al. '98]



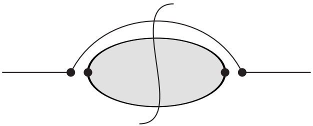
Strategy from Lattice QCD+QED [Bruno et al '26]

- delicate short [Boyle et al' 26] and long distance effects
- shed light given current tensions in e^+e^- panorama

Lattice QCD(+QED): first-principle systematically-improvable approach

- a fully inclusive strategy
- taking care of short-distance renormalization
- discussing non-factorizable effects (aka G_{EM})

$$\tau^-(P) \to \nu(q)\,\mathrm{had}^-_V(p)$$



$$d\Gamma = \frac{1}{4m} d\Phi_q \sum_f d\Phi_f \sum_{\mathrm{spin}} |\mathcal{M}_f|^2$$

Isovector spectral density isospin limit $= \rho$ $\left[d\Phi_q = \frac{d^3q}{(2\pi)^3 2\omega_q} \right]$

$$\frac{d\Gamma}{ds} = \frac{G_F^2 |V_{ud}|^2 m_\tau^3}{8\pi} \kappa(\hat{s})\,\rho(s)\,, \quad \kappa(\hat{s}) = (1+2\hat{s})\,(1-\hat{s})^2 \quad \hat{s} = \frac{s}{m_\tau^2}$$

from experiment we get $\frac{1}{\Gamma}\frac{d\Gamma}{ds} \to \frac{\Gamma}{\Gamma_e}\frac{1}{\Gamma}\frac{d\Gamma}{ds} = \frac{1}{\Gamma_e}\frac{d\Gamma}{ds}$

$$\Gamma_e = \Gamma(\tau \to e\bar{\nu}\nu) = \frac{\mathcal{B}_e \Gamma}{\mathcal{B}} = \frac{G_F^2 m_\tau^5}{192\pi^3}$$

Large uncertainties from ratio of form factors

		Refs. [166, 194]	Ref. [209]	Refs. [237, 247]	Our estimate
Phase space		−7.88	−7.52	–	−7.7(2)
S_{EW}		−12.21(15)	−12.16(15)	–	−12.2(1.3)
G_{EM}		−1.92(90)	$(-1.67)^{+0.60}_{-1.39}$	–	−2.0(1.4)
FSR		4.67(47)	4.62(46)	4.42(4)	4.5(3)
ρ–ω mixing		4.0(4)	2.87(8)	3.79(19)	3.9(3)
$\frac{F_\pi^V}{f_+}$ (w/o ρ–ω)	ΔM_ρ	$0.20(^{+27}_{-19})(9)$	$1.95^{+1.56}_{-1.55}$	–	
	$\Delta\Gamma_\rho(\Delta M_\pi)$	4.09(0)(7)	3.37	–	
	$\Delta\Gamma_\rho(\pi\pi\gamma)$	−5.91(59)(48)	−6.66(73)	–	
	$\Delta\Gamma_\rho(g_{\rho\pi\pi})$	–	–	–	
	Total	−1.62(65)(63)	$(-1.34)^{+1.72}_{-1.71}$	–	−1.5(4.7)
Sum		−14.9(1.9)	$(-15.20)^{+2.26}_{-2.63}$	–	−15.0(5.1)

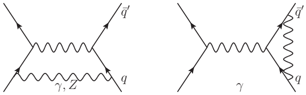
Dispersive representation of radiative corrections [Colangelo et al '25]

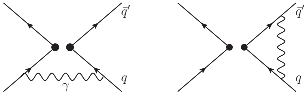
improved estimate $G_{\mathrm{EM}} \to -5.4(5)$

short dist. and rad.cor in “form factors” dominant errors

Hadronic τ decays. Short distance renormalization

Wilson coefficient: matching between SM and EFT at $O(\alpha)$





W-regularization [Sirlin '78][Sirlin '82][Marciano, Sirlin '88][Braaten, Li '90]

- universal UV divergences re-absorbed in G_F
- process-specific corrections in S_{EW}, i.e. Wilson coefficient

Effective Hamiltonian at $O(\alpha)$: $H_W \propto G_\mathrm{F} C^\mathsf{s}(\mu)\mathcal{O}^\mathsf{s}(\mu)$

- finite scheme-dependent terms $O(\alpha)$ must match between C and $\mathcal{O}$

W-regularization [Sirlin '78][Sirlin '82][Marciano, Sirlin '88][Braaten, Li '90]

[Brod, Gorbahn '08][Gorbahn et al '23][Bigi et al '23][Cirigliano et al '23]

$$C^{\overline{\mathrm{MS}}}(\mu) = U(\mu,\mu_W) C^{\overline{\mathrm{MS}}}(\mu_W) \quad \gamma_W = \{\alpha, \alpha\alpha_s\} + O(\alpha\alpha_s^2, \alpha^2)$$

Dim.Reg. not suitable for non-perturbative Lattice QCD
a different scheme needed and a corresponding matching with $\overline{\mathrm{MS}}$

Regularization Invariant Momentum schemes [Martinelli et al '94]
suitable for Lattice QCD = off-shell (massless) quark states
$C^{\overline{\mathrm{MS}}\to\mathrm{RI}}(\mu) = \{1, \alpha, \alpha\alpha_s\} + O(\alpha\alpha_s^2, \alpha^2)$ [Gorbahn et al '23]

Physical effective Hamiltonian $H_W \propto C^{\mathrm{RI}}(\mu) Z^{\mathrm{RI}}(\mu, a) \mathcal{O}^{\mathrm{latt}}(a)$
$C^{\mathrm{RI}}(\mu) = C^{\overline{\mathrm{MS}}\to\mathrm{RI}}(\mu) U(\mu,\mu_W) C^{\overline{\mathrm{MS}}}(\mu_W)$
$\to$ scheme + scale + gauge dependence cancels $C^{\mathrm{RI}} \leftrightarrow Z^{\mathrm{RI}}$

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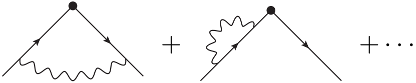
$$C^{\mathrm{RI}}(\mu) = C^{\overline{\mathrm{MS}} \to \mathrm{RI}}(\mu) U(\mu, \mu_W) C^{\overline{\mathrm{MS}}}(\mu_W)$$

$\rightarrow$ scheme + scale + gauge dependence cancels $C^{\mathrm{RI}} \leftrightarrow Z^{\mathrm{RI}}$

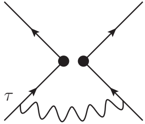
$$Z_{\mathcal{O}}(\mu^2) = \sqrt{Z_\tau(\mu^2)}\, Z_{V,\mathrm{ud}}(\mu^2) Z_{\mathrm{tri}}(\mu^2)$$

$Z_{V,\mathrm{ud}}(\mu) =$ QCD factorized radiative corrections



$\Lambda_\mu =$ quark bilinear amputated GF

four-fermion operator $\Lambda_\mu \otimes \gamma^L_\mu$



$Z^{\mathrm{RI}}_{\mathrm{tri}}(\mu) =$ triangle short-distance radiative corrections

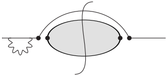
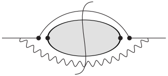
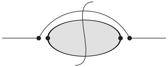
[MB, Boyle et al '26]

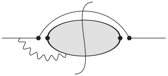
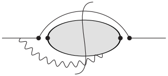
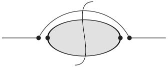
A careful study of [Gorbahn et al '23] led to several minor improvements

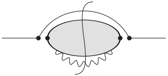
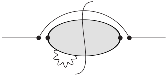
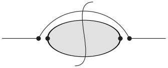
- reconnect w/ previous literature on RI-SMOM renormalization
- definition of simpler convention for our calculation

A strategy for the HVP prediction from τ decays

Three IR safe classes, each w/ separate UV counterterms

 $+$  $+\cdots+(Z_\tau(\mu)-1)$ 

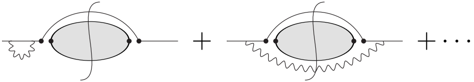
 $+$  $+\cdots+2(Z_{\text{tri}}^{\text{RI}}(\mu)-1)$ 

 $+$  $+\cdots+2(Z_{V,\text{ud}}(\mu)-1)$ 

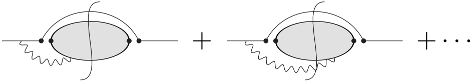
Radiative corrections

Book-keeping

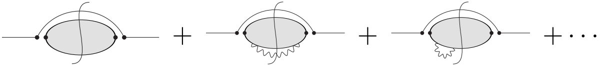
1. Start from experimental data $\left[\frac{1}{\Gamma_e}\frac{d\Gamma}{ds}\right]^{\text{exp}}$
2. subtract initial state radiative corrections (analytic)



3. subtract non-factorizable radiative corrections



4. divide Wilson coefficient and Laplace transform
5. we are left with scale+scheme+gauge dependent

$G^{W,\text{exp}}(t,\mu) \simeq Z^2_{V,\text{ud}}$ 

$[Z_{V,\text{ud}} = 1 + O(\alpha)]$

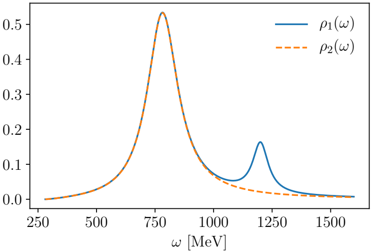
HVP correlator $G^\gamma(t) = -\frac{1}{3}\sum_k \int d^3\boldsymbol{x}\ \langle j_k^\gamma(t,\boldsymbol{x}) j_k^\gamma(0)\rangle = \int d\omega e^{-\omega t}\rho(\omega)$

spectral density $\rho(\omega) \propto R(\omega)$

Lattice correlator is smeared=folded R-ratio

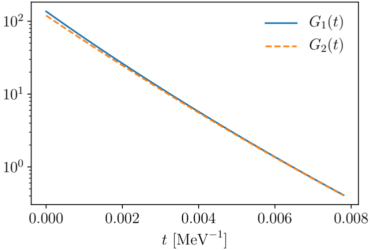
Minkowski

$\rho_i(\omega)$ time-like cross sections



Euclidean

$G_i(t) = \int d\omega\, e^{-\omega|t|}\rho_i(\omega)$



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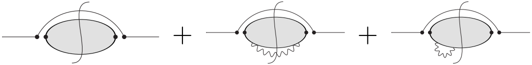
Lattice correlator is smeared=folded R-ratio

Charged correlator $G^W(t,\mu) = [Z_{\rm ud}^{\rm RI}(\mu)]^2 \langle j^+(t,\mathbf{0}) j^-(0)\rangle_{{\rm QCD}(1+O(\alpha))}$

$G^{W,{\rm exp}}(t,\mu) \simeq Z^2_{V,{\rm ud}}$  $+\cdots$

$[Z_{V,{\rm ud}} = 1 + O(\alpha)]$

Lattice G^W is "smeared τ data w/o initial state and w/o triangle effects"

Neutral correlator $G^\gamma(t) \equiv -\frac{1}{3}\sum_k \int d^3\boldsymbol{x}\ \langle j_k^\gamma(t,\boldsymbol{x}) j_k^\gamma(0)\rangle$

Isospin decomposition $G^\gamma(t) \equiv G_{00}^\gamma(t) + 2G_{01}^\gamma(t) + G_{11}^\gamma(t)$

Charged correlator $G^W(t,\mu) = [Z_{\mathrm{ud}}^{\mathrm{RI}}(\mu)]^2 \langle j^+(t,\mathbf{0}) j^-(0)\rangle_{\mathrm{QCD}(1+O(\alpha))}$

charged vector current $j_\mu^-(x) = \frac{1}{\sqrt{2}}\bar{u}(x)\gamma_\mu d(x)$

$G^\gamma(t) = G_{00}^\gamma(t) + 2G_{01}^\gamma(t)$		Lattice QCD+QED
$+ G_{11}^\gamma(t) - G^W(t,\mu)$	$= \delta G_{11}(t,\mu)$	Lattice QCD+QED
$+ G^W(t,\mu) - G^{W,\mathrm{exp}}(t,\mu)$		Exp. data or theory
$+ G^{W,\mathrm{exp}}(t,\mu)$		Exp. data and theory

Calculate a_μ, windows etc... of any of the above

e.g. $\int dt\, w(t,m_\mu)\, G^\gamma(t)\, \Theta^{\mathrm{win}}(t)$

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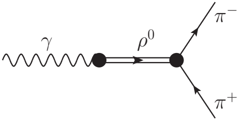
$$
\begin{aligned}
G^\gamma(t) = \quad & G_{00}^\gamma(t) + 2G_{01}^\gamma(t) && && \text{Lattice QCD+QED} \\
& + G_{11}^\gamma(t) - G^W(t,\mu) && = \delta G_{11}(t,\mu) && \text{Lattice QCD+QED} \\
& + G^W(t,\mu) - G^{W,\exp}(t,\mu) && && \text{Exp. data or theory} \\
& + G^{W,\exp}(t,\mu) && && \text{Exp. data and theory}
\end{aligned}
$$

Calculate a_μ, windows etc... of any of the above

e.g. $\int dt\, w(t,m_\mu)\, G^\gamma(t)\, \Theta^{\mathrm{win}}(t)$

A rough sketch of where various effects are hidden

$\delta G_{11} = O(\alpha) \quad \leftrightarrow$



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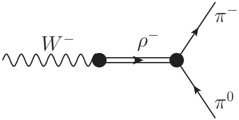
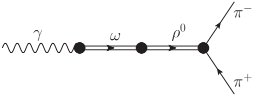


$m_{\rho^0} \neq m_{\rho^\pm}$, $\Gamma_{\rho^0} \neq \Gamma_{\rho^\pm}$, $m_{\pi^0} \neq m_{\pi^\pm}$

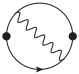
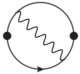
$G_{01} \leftrightarrow \quad \delta_{\rho\omega} \simeq O(m_{\rm u} - m_{\rm d}) + O(e^2)$

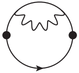
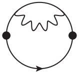


$$\delta G_{11}(t,\mu) = G_{11}^{\gamma}(t) - G^{W}(t,\mu) = O(\alpha)$$

- calculable in Euclidean space from Lattice QCD
- fully inclusive both in channels and energy
- scheme + scale + gauge dependent
- large cancellations, all diagrams cancel out except for V and F

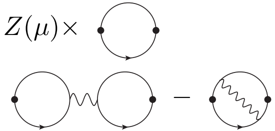
$\frac{1}{2}[\bar{u}u - \bar{d}d]$ $\qquad$ $\bar{u}d$ $\qquad$ = Total

$\frac{1}{2}Q_{\mathrm{u}}^2 + \frac{1}{2}Q_{\mathrm{d}}^2$  $\qquad$ $-Q_{\mathrm{u}}Q_{\mathrm{d}}$  $\qquad$ $\to (Q_{\mathrm{u}} - Q_{\mathrm{d}})^2$

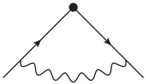
$Q_{\mathrm{u}}^2 + Q_{\mathrm{d}}^2$  $\qquad$ $-Q_{\mathrm{u}}^2 - Q_{\mathrm{d}}^2$  $\qquad$ $\to 0$

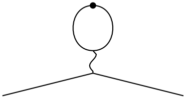
All insertions of dressed propagators cancel out

$$\delta G_{11}(t,\mu) = Z_V^{\mathrm{QCD}}(Z_{\gamma,1} - Z_{V,\mathrm{ud}}(\mu^2))(c)(t)$$
$$+e^2\frac{(Q_\mathrm{u}-Q_\mathrm{d})^2}{4}(Z_V^{\mathrm{QCD}})^4((F)-(V))(t)$$



Renormalization of δG_{11} proceeds via two topologies
gauge inv. of neutral current simplifies renorm. of charged current

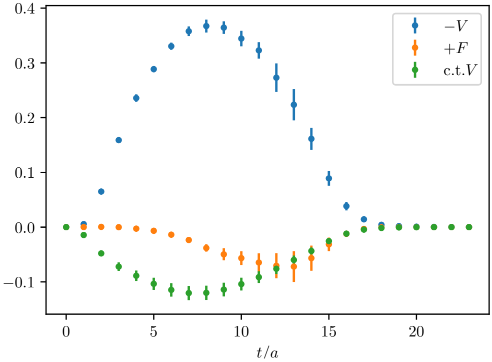




Neutral vs charged correlators

Preliminary

[Counter-term for V] + [$F - V$] (counter-term for F ongoing)



blinded analysis

$m_\pi \simeq 280$ MeV,
$a^{-1} = 1.73$ GeV

fourth moment t^4 with smooth cutoff at ≈ 1.6 fm

counter-term at $\mu = m_\tau$ (twisted BC)

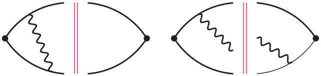
Lattice syst. errs → finite-volume and discretization errors
we plan two volumes and two lattice spacings at phys. pion mass

NEUTRAL VS CHARGED CORRELATORS

Systematic errors

G^W calculated from Lattice QCD+QED differs from $G^{W,\mathrm{exp}}$

syst effect $\delta G^W = G^W - G^{W,\mathrm{exp}} = O(\alpha^0) + O(\alpha^1)$

$\delta G_{11}(t,\mu) \to$ 

lattice fully inclusive vs experiment $\theta(m_\tau - \sqrt{s})$
- vector spectrum $\sqrt{s} > m_\tau$ at $O(\alpha^0)$
- vector spectrum $\sqrt{s} > m_\tau$ also has $O(\alpha)$
- suppressed in intermediate and long-dist windows

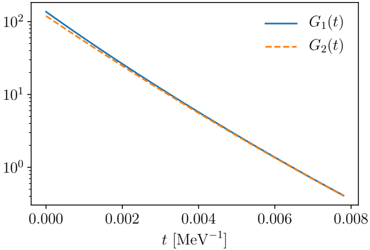
mixing between axial and vector states
- $A \to V + \gamma$ inside exp.data, known effect
  - estimated small in EFT [Cirigliano et al '01][Colangelo et al' 25]
- $V \to A + \gamma$ inside lattice data, must be taken in account

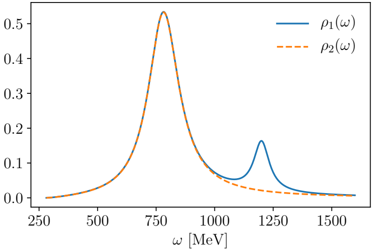
What are inverse problems and why do they matter here?

Lattice correlator
$\langle \widetilde{J}(t)\, \widetilde{J}(0) \rangle = \int d\omega\, e^{-\omega t}\, \rho(\omega)$

Inverse Laplace
$[e^{-\omega t}] \to [\kappa(\omega)]$

Physical observable
$\rho_\kappa = \int d\omega\, \kappa(\omega)\, \rho(\omega)$



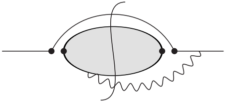


$f(t)$ such that $\rho_\kappa = \int dt\, f(t)\, \langle \widetilde{J}(t)\, \widetilde{J}(0) \rangle$: is it possible?

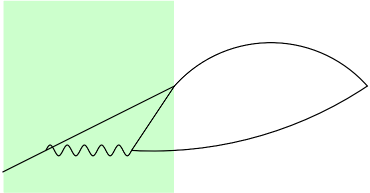
Non-factorizable corrections

Time orderings

$$\int_x e^{ikx} \langle 0|\mathcal{J}^{+}_{\mu}(0)\, \delta(\hat{H} - \sqrt{s})\, T\{\widetilde{\mathcal{J}}^{\gamma}_{\alpha}(x)\mathcal{J}^{L,-}_{\nu}(0)\}|0\rangle$$



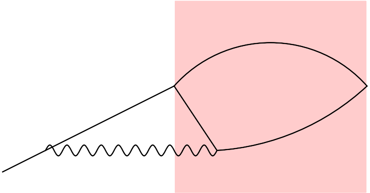
integral $x_0 < 0$



$\to$ directly calculable from Lattice QCD

integral ds
$\to$ calculable via exact reconstruction tail

integral over $x_0 > 0$ and ds



$\to$ $\boldsymbol{k} = 0$ calculable via extract reconstruction tail
$\to$ $|\boldsymbol{k}| > 0$ double inverse problem

[Hansen-Bulava '19][Bruno-Hansen '20]

Amplitudes from infinite-volume correlators via inverse Laplace

i. $\langle 0|j_\mu(t)\pi_{-\boldsymbol{p}}(0)|\pi,\boldsymbol{p}\rangle = \langle 0|j_\mu(0)e^{-\hat{H}t}\pi_{-\boldsymbol{p}}(0)|\pi,\boldsymbol{p}\rangle$

ii. replace $e^{-\hat{H}t} \to$ Cauchy pole

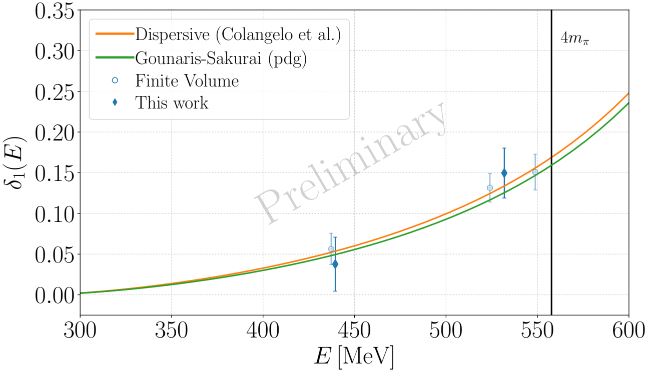
$$\langle 0|j_\mu(t)\pi_{-\boldsymbol{p}}(0)|\pi,\boldsymbol{p}\rangle \to \mathrm{inv.lap} \to \langle 0|j_\mu(0)\frac{1}{\hat{H}-E-i\varepsilon}\pi_{-\boldsymbol{p}}(0)|\pi,\boldsymbol{p}\rangle$$

iii. split in real and imaginary parts and take $\varepsilon \to 0$, $\tan\delta(s) = \dfrac{\mathrm{Im}F_\pi(s)}{\mathrm{Re}F_\pi(s)}$

iv. remove LSZ $\sqrt{Z} = \langle 0|\pi_{-\boldsymbol{p}}(0)|\pi,\boldsymbol{p}\rangle$ factor and we have the amplitude!

delicate limit $\varepsilon \to 0$ only after $L \to \infty$

[G. Morandi Lattice '26]



first validation of the method

bypasses limits of Lüscher finite-volume approach: valid at $\sqrt{s} > 4m_\pi$

Strategy is now complete [Bruno et al '26]

- short-distance in RI-SMOM scheme
  - first results renormalization of charged current
- initial-state corrections analytic
- final-state radiation in Euclidean space
- first results for spectrum above m_τ [J Parrino's talk]
- non-factorizable still pose severe inverse problem
  - sinergy with dispersive approach likely in short run

Thanks for your attention

Preserving Ward Identity in QCD bilinear very important

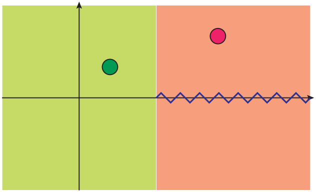
- prevents $C(\mu) = 1 + O(\alpha_s^n) + \cdots$ [Gorbahn et al '23]
- simpler assessment of syst errors
- $C^{\mathrm{RI}}(\mu)$ calculated w/ single-trace convention in [Gorbahn et al '23]

[MB, Boyle et al '26]

1. Extension of RI-SMOM scheme [Sturm et al '08] in pure QCD
   $P_\mu^{\mathrm{RI/SMOM}_y} = \frac{1}{2N_c p^2}(y\not{p} + (y-1)\not{p}')q_\mu$
2. Natural embedding in semileptonic operator
   double trace convention $\Lambda_\mu \otimes \gamma_\mu$
3. equivalent to [Gorbahn et al '23] via Fierz identities

$$\kappa(\omega) = \frac{1}{\omega - \bar\omega + i\sigma} \to f(t) \text{ such that } \rho_\kappa = \int dt\, f(t)\, C(t)?$$

$$\rho_\kappa = \lim_{T\to\infty} \int d\omega \int_0^T dt\, e^{-(\omega-\bar\omega-i\sigma)t}\,\rho(\omega) = \lim_{T\to\infty} \int_0^T dt\, e^{(\bar\omega+i\sigma)t}\, C(t)$$



Limit $T \to \infty$ exists if $\bar\omega$ below support of ρ

interplay between pole $\bar\omega + i\sigma$ vs branch cut

ρ has branch cut starting at multi-particle thresholds $E_{\rm thr}$

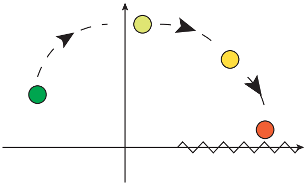
if $\bar\omega \leq E_{\rm thr}$ $\quad \to \forall t\ \exists M > 0 \mid f(t)C(t) < e^{-Mt}$
e.g. HVP contribution to $(g-2)_\mu$ [Blum '02][Bernecker-Meyer '11]

if $\bar\omega > E_{\rm thr}$ $\quad \to$ inverse problem, no "direct" analytic continuation

Increasingly high attention from the community (formal and theoretical)
[Backus, Gilbert '68][Hansen, Meyer, Robaina '17][Hansen, Lupo, Tantalo '19]
[Bailas et al '20][Del Debbio et al '24][..][Jay et al '24][..][MB, Giusti, Saccardi '25][..]
[Bresciani, MB, Hansen '26]

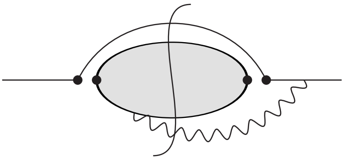
several calculations mostly by [ETMC]

HVP long-distance reconstruction = direct solution of inv.-lap
[RBC/UKQCD '24][Mainz '24][FNAL/HPQCD/MILC '24]



inverse problem solvable w/ modern tech
but closer to the cut more challenging
$\rightarrow$ noise, finite-volume effects ...

1. ampl $\mathcal{M}_f^{\mathrm{real,had}}$ from time-ordered prod
   $\int_x e^{ikx} \langle V| T\{\mathcal{J}_\alpha^\gamma(x)\mathcal{J}_\mu^{L,-}(0)\}|0\rangle$



2. interference $\mathcal{M}_f^{\mathrm{real,had}} \left[\mathcal{M}_f^{\mathrm{real},\tau}\right]^\dagger$
   $\int_x e^{ikx} \langle 0|\mathcal{J}_\mu^+(0)|V\rangle\langle V|T\{\mathcal{J}_\alpha^\gamma(x)\mathcal{J}_\nu^{L,-}(0)\}|0\rangle$

3. hadronic phase-space + sum over all vector states
   $$\int_x e^{ikx} \langle 0|\mathcal{J}_\mu^+(0)\, \delta(\hat{H}-\sqrt{s})\, T\{\widetilde{\mathcal{J}}_\alpha^\gamma(x)\mathcal{J}_\nu^{L,-}(0)\}|0\rangle$$

4. calculable from Euclidean correlator $\langle \widetilde{j}_\mu^+(t,\mathbf{0})\, \widetilde{j}_\alpha^\gamma(\tau,\boldsymbol{k})\, j_\nu^{L,-}(0)\rangle$?
   there is δ-function $\to$ first inverse problem?
   there is T product $\to$ second inverse problem?

Notation: $\mathcal{J}^{L,-}$ charged weak, $\mathcal{J}^+$ charged vector, $\mathcal{J}$: Minkowski
$\widetilde{j}$: Fourier trafo