Spurion Matching of HVP and HLbL Diagrams

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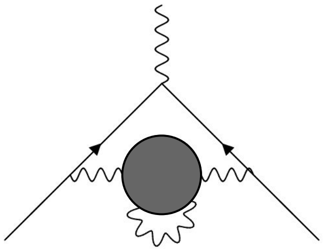
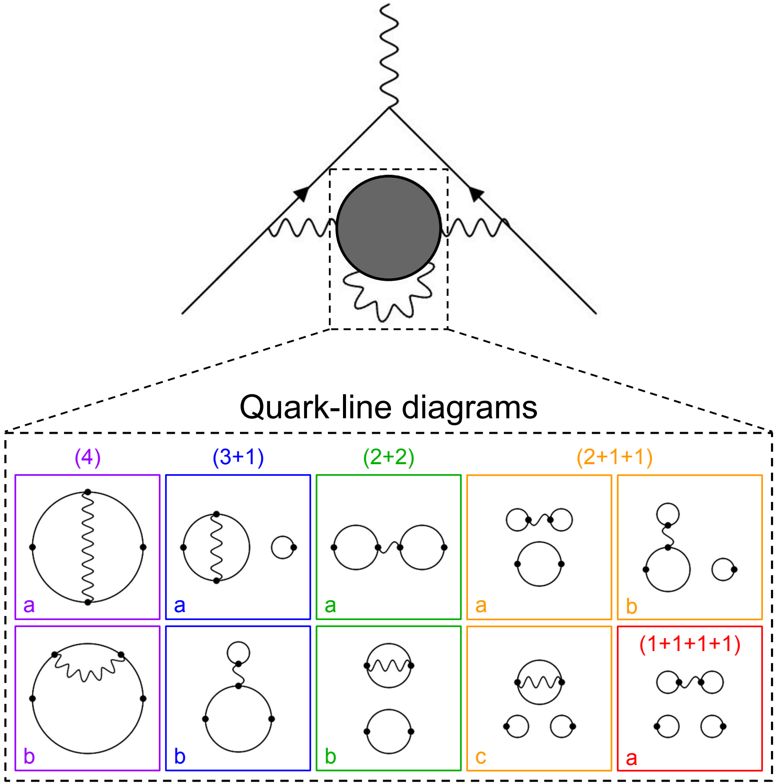
University of Colorado Boulder



Background and Motivation

QED Corrections to LO HVP

- $O(\alpha^3)$ QED corrections to $O(\alpha^2)$ pure-QCD LO HVP
- Needed for sub-percent HVP from lattice QCD
- Some topologies are more difficult than others
- EFTs (χPT) can give insight into (LD-dominant) QED corrections



QED Corrections in χPT

$$\delta a_\mu^{\mathrm{QED},\chi\mathrm{PT}} = \overbrace{a_\mu^{\pi^0} + a_\mu^{\eta} + a_\mu^{\eta'}}^{\text{Neutral Exchange}} + \overbrace{a_\mu^{\pi^+\pi^-} + a_\mu^{K^+K^-}}^{\text{Charged Loop}} + \cdots$$

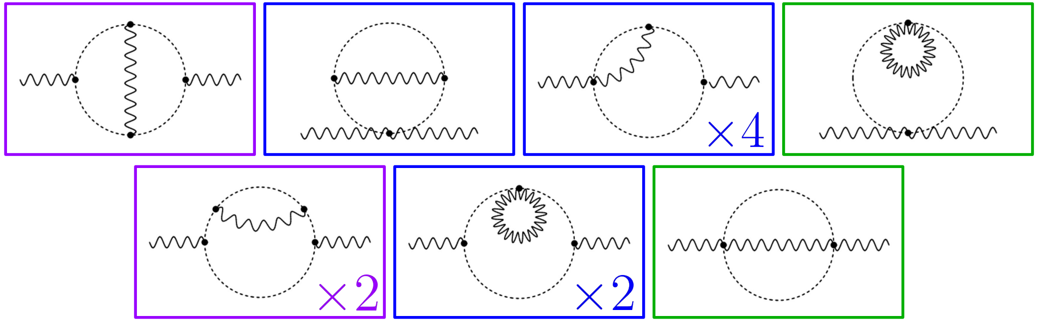
Neutral Pseudoscalar Exchange

- Neutral pseudoscalars enter χPT via Wess-Zumino-Witten term

$$\mathcal{L}_{\mathrm{WZW}} \supset -\frac{N_c \alpha}{12\pi}\pi^0 F\widetilde{F}$$

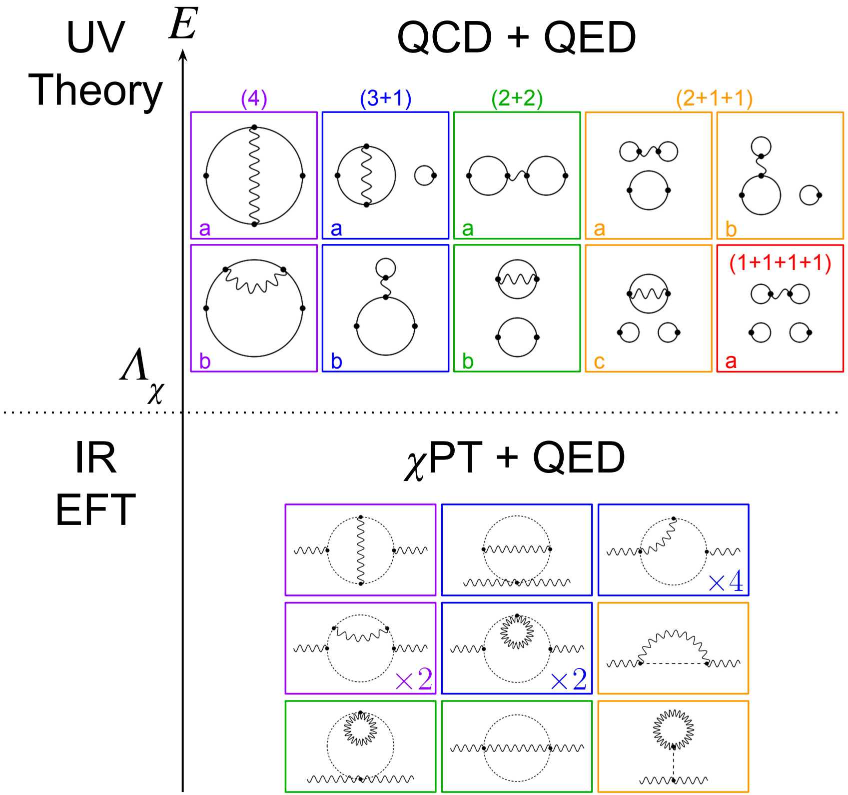
- η' included in Large-N_c χPT

Charge Pseudoscalar Loop



- Charged pseudoscalar contributions equivalently calculated in scalar QED

* Can also include transition form factors (e.g., VMD)



What

- Relate objects (diagrams, operators, …) between UV theory and IR EFT

Why

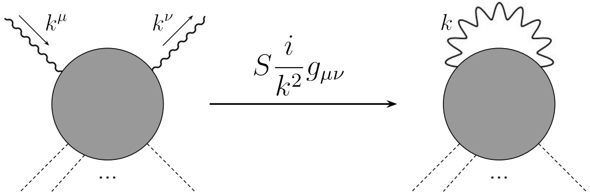
- Predict individual UV diagrams at low E with EFT to supplement lattice analysis

How

- Exploit structure (symmetries, anomalies, …) present across UV and IR scales
- Diagram separation requires unphysical modification of the theory

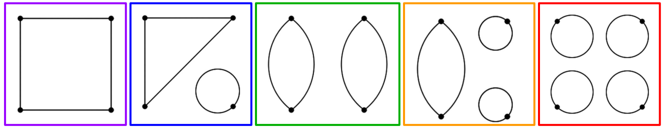
Relation to LbL

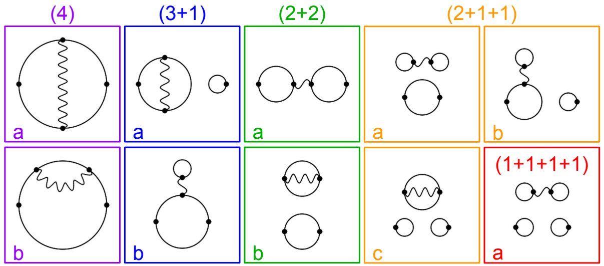
- QED corrections to VP related to LbL by contracting pair of EM currents
- Matching goes through proportionally



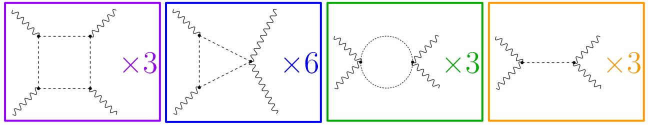
Biloshytskyi *et al*. JHEP [2209.02149]
Parrino *et al*. JHEP [2501.03192]

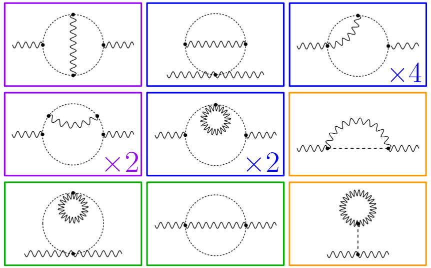
UV





IR





LbL

VP + LO QED

Matching via Spurions

Matching via Spurions

- Explicitly broken symmetries can be restored by promoting parameters of Lagrangian to fields called “spurions”
  - E.g., chiral symmetry breaking by mass matrix: $M \rightarrow U_R M U_L^\dagger$
  - Bookkeeping trick to utilize algebraic structure of broken symmetry
  - Spurions set to physical values at the end
- Background fields can also be treated as spurions

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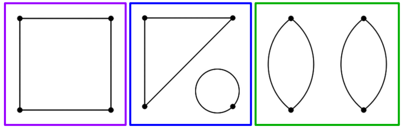
- Background fields can also be treated as spurions

- For VP/LbL matching, restore $\mathrm{SU}(N_f)_V$ QED isospin symmetry with spurions
  - Charge matrix spurion $Q \rightarrow U_V Q U_V^\dagger$ $\xleftrightarrow{\text{Equivalent}}$
  - Non-abelian background spurion $ieQA^\mu \rightarrow ieB^\mu = ieB^\mu_\alpha T^\alpha$

- "Unphysical" step: Q not fixed to physical quark charges

Outline of Spurion Matching Procedure

1. UV diagrams & current correlators



$C_q^{(4)}$ $D_{qq'}^{(3+1)}$ $D_{qq'}^{(2+2)}$

$$\cdots \longleftrightarrow Q_q Q_{q'} \cdots \langle J_q J_{q'} \cdots \rangle_{\mathrm{UV}}$$

Quark Basis: $J_q = J_u, J_d, \cdots$

Isospin Basis: $J_\alpha = J_0, J_3, J_8, \ldots$

Detailed example in backup

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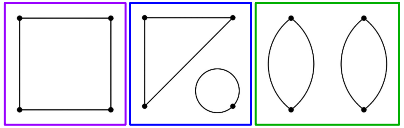

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2. Connected generating functionals (w.r.t. spurions)

$$\frac{\delta^n W_{\mathrm{UV}}}{\delta[ieB_\alpha^\mu]\cdots} \sim \langle J_\alpha^\mu \cdots \rangle_{\mathrm{UV}} \xleftrightarrow{\text{Matching}} \frac{\delta^n W_{\mathrm{IR}}}{\delta[ieB_\alpha]\cdots} \sim \langle J_\alpha \cdots \rangle_{\mathrm{IR}} + \left\langle \frac{\delta J_\alpha}{\delta[ieB_\beta]} \cdots \right\rangle_{\mathrm{IR}} + \cdots$$

* “Connected” means “gluon connected” in quark diagrams

Detailed example in backup

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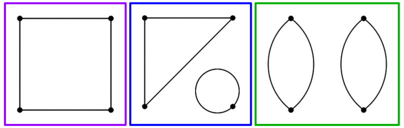

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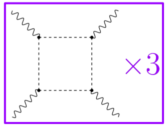
3. IR diagrams $\langle J_\alpha \cdots \rangle_{\mathrm{IR}} + \left\langle \frac{\delta J_\alpha}{\delta[ieB_\beta]} \cdots \right\rangle_{\mathrm{IR}} + \cdots \longleftrightarrow$



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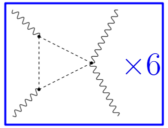

$+ \cdots$

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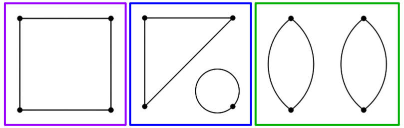

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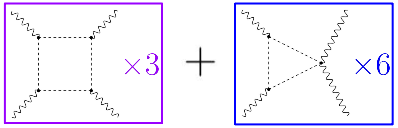
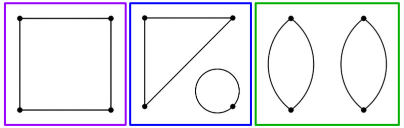
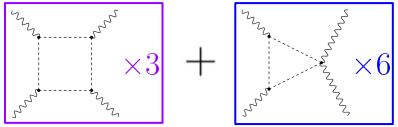


$+\cdots$

4. Matching relations



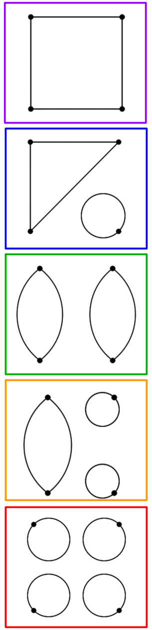
$\cdots \longleftrightarrow$



$+\cdots$

Detailed example in backup

N_f = 2 + 1 LbL Matching Relations



$C_q^{(4)}$

$D_{qq'}^{(3+1)}$

$D_{qq'}^{(2+2)}$

≈ 0

≈ 0

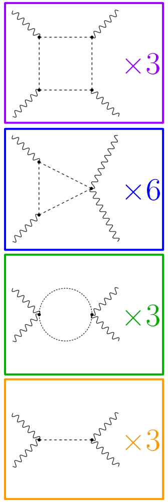
$$\overbrace{\qquad\qquad\qquad\qquad}^{\text{UV}}\overbrace{\qquad\qquad\qquad\qquad}^{\text{IR}}$$

$$\begin{aligned}
C_l^{(4)} &= \frac{17}{81}\left[\,2\mathcal{I}_{\pi^+} + \mathcal{I}_{K^+} + 2\mathcal{I}_{\pi^0}\right] \\
D_{ll}^{(3+1)} &= \frac{7}{81}\left[-4\mathcal{I}_{\pi^+}\right] \\
D_{ls}^{(3+1)} &= -\frac{7}{81}\left[-4\mathcal{I}_{K^+}\right] \\
D_{sl}^{(3+1)} &= -\frac{1}{81}\left[-4\mathcal{I}_{K^+}\right] \\
D_{ll}^{(2+2)} &= \frac{25}{81}\left[\,3\mathcal{I}_{\pi^+} - \mathcal{I}_{\pi^0} + \mathcal{I}_{\eta} + \mathcal{I}_{\eta'}\right] \\
2D_{ls}^{(2+2)} &= \frac{5}{81}\left[+6\mathcal{I}_{K^+} - 4\mathcal{I}_{\eta} + 2\mathcal{I}_{\eta'}\right] \\
C_s^{(4)} + D_{ss}^{(3+1)} + D_{ss}^{(2+2)} &= \frac{1}{81}\left[+2\mathcal{I}_{K^+} + 4\mathcal{I}_{\eta} + \mathcal{I}_{\eta'}\right]
\end{aligned}$$

$$\mathcal{I}_P \equiv \sum \begin{pmatrix}\text{IR diagrams} \\ \text{with } P\end{pmatrix}$$

$\mathcal{I}_{\pi^+}, \mathcal{I}_{K^+}$

$\mathcal{I}_{\pi^0}, \mathcal{I}_{\eta}, \mathcal{I}_{\eta'}$



- Spurion matching gives maximal flavor separation, and partial topological separation
- Straightforward to apply to VP (or any such n-pt fcn) or $N_f \neq 2 + 1$

η-η' mixing not shown for simplicity (see backup)

Comparison to Partial Quenching

See backup for refs to other related procedures

Matching via Partial Quenching

VP

Della Morte & Jüttner JHEP [1009.3783]

LbL

Chao *et al*. EPJC [2006.16224, 2104.02632]

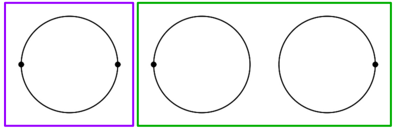
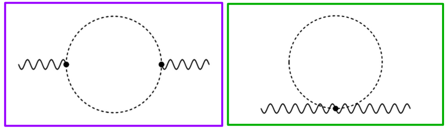
- Add N_v valence flavors (and ghosts)
- Isospin symmetry corresponds to $\mathrm{SU}(N_f + N_v \mid N_v)$ graded Lie group
- Valence flavors limit possible Wick contractions, e.g., LO VP:

$$\begin{aligned} C_{\mathrm{QCD}}(y,x) &= \sum_{q=u,d,s,\ldots} \langle \bar{q}(y)\Gamma' q_v(y)\bar{q}_v(x)\Gamma q(x)\rangle_{\mathrm{PQQCD}} \\ &\quad + \sum_{q=u,d,s,\ldots} \langle \bar{q}(y)\Gamma' q(y)\bar{q}'_v(x)\Gamma q_v(x)\rangle_{\mathrm{PQQCD}} \\ &\equiv C^{\mathrm{Conn}}_{\mathrm{PQQCD}}(y,x) + C^{\mathrm{Disc}}_{\mathrm{PQQCD}}(y,x)\,. \end{aligned}$$

- Can match partially quenched flavors from UV (PQQCD) to IR (PQχPT)
- “Unphysical” step: $\mathrm{SU}(N_f + N_v \mid N_v)$ isospin symmetry

Comparison Example: N_f = 2 + 1 LO VP

UV | IR

$C_q^{(2)}$  $D_{qq'}^{(1+1)}$ $\longleftrightarrow$ $\mathcal{I}_{\pi^+}, \mathcal{I}_{K^+} = \sum$ 

Matching via Spurions

$$
\begin{aligned}
C_l^{(2)} &= \frac{5}{9}\left[\, 2\mathcal{I}_{\pi^+} + \mathcal{I}_{K^+} \,\right] \\
D_{ll}^{(1+1)} &= \frac{1}{9}\left[-\mathcal{I}_{\pi^+} \qquad\qquad \right] \\
2D_{ls}^{(1+1)} &= -\frac{1}{9}\left[\qquad\qquad - 2\mathcal{I}_{K^+} \right] \\
C_s^{(2)} + D_{ss}^{(1+1)} &= \frac{2}{9}\left[\qquad\qquad 2\mathcal{I}_{K^+} \,\right]
\end{aligned}
$$

- Maximal flavor (l,s) separation
- Partial topological ((2), (1+1)) sep.
- Nothing fictitious (with strong isosym)

Matching via PQing [1009.3783]

$$
C_l^{(2)} + C_s^{(2)} = \frac{10}{9}\mathcal{I}_{\pi^+} + \frac{7}{9}\mathcal{I}_{K^+} + \frac{1}{9}\mathcal{I}_{\bar{s}s}
$$

$$
D_{ll}^{(1+1)} + 2D_{ls}^{(1+1)} + D_{ss}^{(1+1)} = -\frac{1}{9}\mathcal{I}_{\pi^+} + \frac{2}{9}\mathcal{I}_{K^+} - \frac{1}{9}\mathcal{I}_{\bar{s}s}
$$

- No flavor (l,s) separation
- Maximal topological ((2), (1+1)) sep.
- Fictitious $\bar{s}s$ meson

> Both agree in overlapping predictions (sums, strong isospin limits, …)

Conclusions

Conclusions

Summary

- Precision physics from Lattice QCD requires QED corrections
- χPT is useful for understanding low E QED corrections
- Matching gives info on individual UV diagrams from IR EFT
- Charge matrix spurion matching gives maximal flavor separation and partial topological separation
- Spurion approach gives different info than PQ (agreement in overlapping predictions)

Future Work

- PQ charge matrix spurions (ongoing)
  - LbL strange topologies
  - VP SIB corrections
  - SIB × QED
- Extension to other EW observables
  - $\Delta\alpha$ and $\sin^2\theta_W$
  - Inclusive τ decay (axial vector sources)
  - Smeared R-ratio

Backup

Relevant and Recent References

Spurions

- Craig TASI 25 “The Formal Frontiers of Particle Physics”
- Song, Sun, Yu [2501.09787]

Isospin/Charge Matching

- Jüttner, Della Morte PoS LAT [0910.3755]
- Gérardin *et al*. PRD [1712.00421]
- Lehner, Parrino, Völklein PRD [2508.21685]

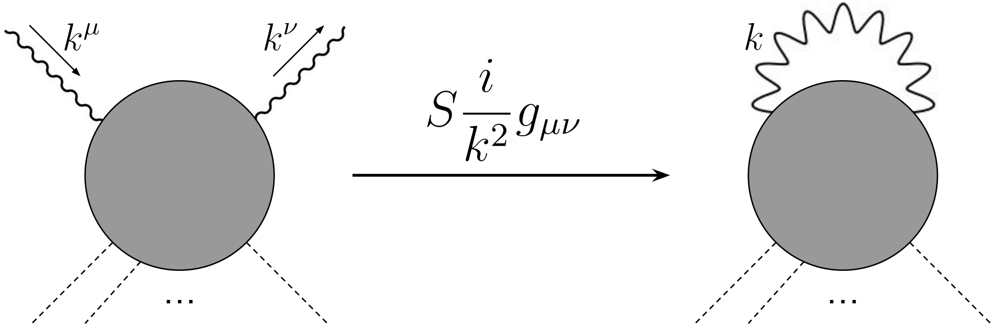
Partially Quenched Matching

- Della Morte & Jüttner JHEP [1009.3783]
- Chao *et al*. EPJC [2006.16224]
- Chao *et al*. EPJC [2104.02632]

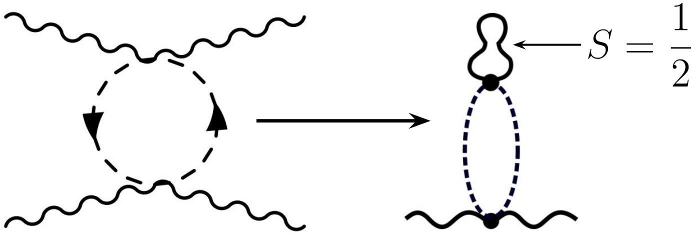
Other

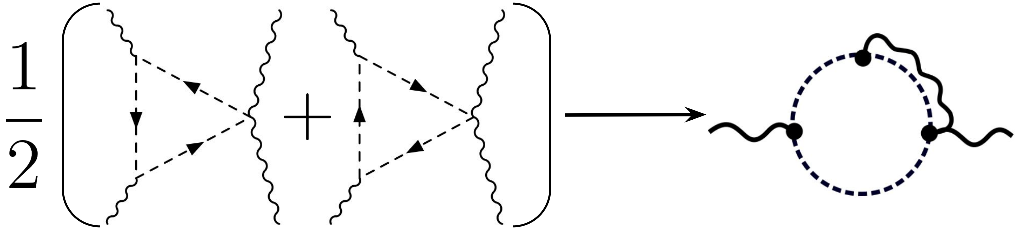
- Biloshytskyi *et al*. JHEP [2209.02149]
- Djukanovic *et al*. JHEP [2411.07969]
- Parrino *et al*. JHEP [2501.03192]

Cottingham Rule



- Can derive n-loop amplitudes from $(n - 1)$-loop
- A rule not a theorem (to my knowledge) ⇒ must be verified (e.g., compare to n-loop amplitudes)
- Useful for doing integrals, e.g., when $(n - 1)$ = 1-loop packages can be used
- Simplifies connected generating functional (only external photons)
- Combinatorics → symmetry factors





η-η' Mixing

- Mass matrix (thus propagator) non-diagonal
- Naive matching gives flavor eigenstates ⇒ must rotate to mass eigenstates
- To leading order, mixing described by single parameter θ (cf. [1612.05473])

$$\eta_{\text{flav}} = \begin{pmatrix} \eta_8 \\ \eta_1 \end{pmatrix} = R^T \begin{pmatrix} \eta \\ \eta' \end{pmatrix} = R^T \eta_{\text{mass}}, \qquad R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

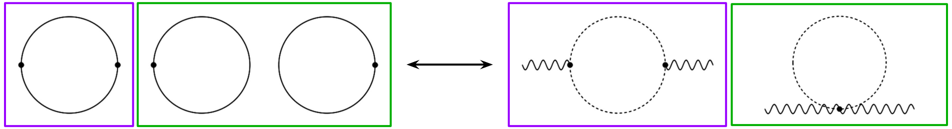
$$M^2_{\text{flav}} = \begin{pmatrix} M^2_{88} & M^2_{81} \\ M^2_{81} & M^2_{11} \end{pmatrix} = R^T \begin{pmatrix} M^2_{\eta} & 0 \\ 0 & M^2_{\eta'} \end{pmatrix} R = R^T M^2_{\text{mass}} R$$

$$S^{\alpha\beta}_{\text{flav}}(p) = \sum_{P=\eta,\eta'} (R^T)^{\alpha P} R^{P\beta} S^P_{\text{mass}}(p)$$

- Matching coefficients for η-η' modified to θ-dependent functions
- Mixing ~10–30% correction (depending on θ)

Example: N_f = 2 + 1 LO VP

$C_q^{(2)}$ $\quad D_{qq'}^{(1+1)}$ $\quad \mathcal{I}_P \equiv \sum \begin{pmatrix}\text{IR diagrams} \\ \text{with } P\end{pmatrix}$



Matching via Spurions

$$
\begin{aligned}
C_l^{(2)} &= (Q_u^2 + Q_d^2)\, X_l = \frac{5}{9}\left[2\mathcal{I}_{\pi^+} + \left(\sin^2\varphi \mathcal{I}_{K^+} + \cos^2\varphi \mathcal{I}_{K^0}\right)\right] \\
D_{ll}^{(1+1)} &= (Q_u + Q_d)^2\, \Delta_{ll} = \frac{1}{9}\left[-\mathcal{I}_{\pi^+}\right] \\
2D_{ls}^{(1+1)} &= 2Q_s\,(Q_u + Q_d)\,\Delta_{ls} = -\frac{1}{9}\left[-2\left(\sin^2\varphi \mathcal{I}_{K^+} + \cos^2\varphi \mathcal{I}_{K^0}\right)\right] \\
C_s^{(2)} + D_{ss}^{(1+1)} &= Q_s^2 X_s = \frac{2}{9}\left[2\left(\sin^2\varphi \mathcal{I}_{K^+} + \cos^2\varphi \mathcal{I}_{K^0}\right)\right]
\end{aligned}
$$

- Maximal flavor (l,s) separation
- Partial topological ((2), (1+1)) sep.
- Fictitious “charged neutral kaons”

Quark ↔Isospin Bases Equations

$$X_l = 2\left\langle J_3^\mu J_3^\nu \right\rangle \qquad 2\Delta_{ls} = -\frac{4}{3}\left\langle J_8^\mu J_8^\nu \right\rangle$$

$$\Delta_{ll} = -\left\langle J_3^\mu J_3^\nu \right\rangle + \frac{1}{3}\left\langle J_8^\mu J_8^\nu \right\rangle \qquad X_s + \Delta_{ss} = \frac{4}{3}\left\langle J_8^\mu J_8^\nu \right\rangle$$

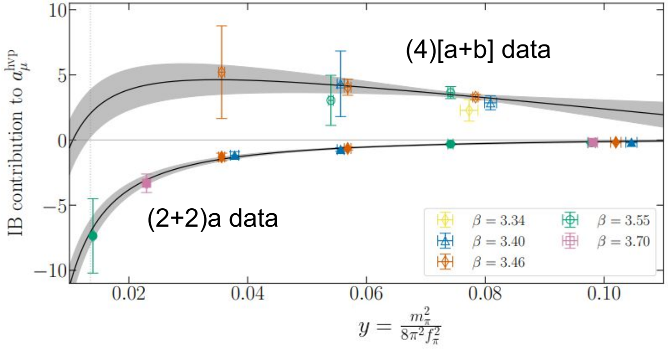
Str isosym ⇒ $\boxed{0 = \left\langle J_3^\mu J_8^\nu \right\rangle + \left\langle J_8^\mu J_3^\nu \right\rangle \Leftrightarrow \mathcal{I}_{K^+}^{\text{VP}} = \mathcal{I}_{K^0}^{\text{VP}}}$

- Strong isospin $\mathrm{SU}(2)_f$ implies K^+ and K^0 diagrams equal (because only flavor dependence though $m_{K^+} = m_{K^0}$)
- Equivalently, can rotate between $K^+ \leftrightarrow K^0$ (i.e., $u \leftrightarrow d$)
- X,Δ don’t depend on QED charges, so are unaffected by QED isospin-breaking effects

- “Zeroing out” K^0’s instead gives wrong total
- Without strong isospin, K^0’s cancel in total (but not individual contributions)

Joint Chiral Ansatz of QED Corrections for Lattice Data

Djukanovic *et al*. JHEP [2411.07969]



- Joint ansatz (4) & (2+2)*a*: Shared pion mass term from PQ matching

$$a_\mu^{\mathrm{hvp}1\gamma^*,(4)} = \frac{34}{81}\frac{A}{m_\pi^3} + b m_\pi^2 + c + 0.22\,\log\frac{m_V^2}{m_\pi^2},$$

$$a_\mu^{\mathrm{hvp}1\gamma^*,(2+2)\mathrm{a}} = \frac{50}{81}\frac{A}{m_\pi^3} + d\,,$$

- Total (w/ (3+1))

$$a_\mu^{\mathrm{hvp}1\gamma^*} = \frac{A}{m_\pi^3} + b m_\pi^2 + c + d + 0.22\,\log\frac{m_V^2}{m_\pi^2}.$$

Analogous spurion ansatz:

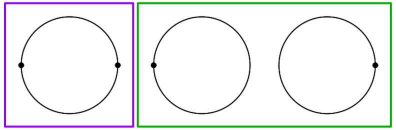
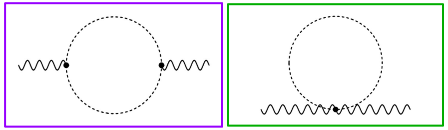
$$\begin{aligned}
a_\mu^{(4)\text{-}l} &= \frac{17}{81}\Big[\,2A_1 m_{\pi^+}^{n_1} + A_2 m_{K^+}^{n_2} + 2A_3 m_{\pi^0}^{n_3}\,\Big]\\
a_\mu^{(3+1)\text{-}ll} &= \frac{7}{81}\Big[-4A_1 m_{\pi^+}^{n_1}\Big]\\
a_\mu^{(3+1)\text{-}ls} &= -\frac{7}{81}\Big[-4A_2 m_{K^+}^{n_2}\Big]\\
a_\mu^{(3+1)\text{-}sl} &= -\frac{1}{81}\Big[-4A_2 m_{K^+}^{n_2}\Big]\\
a_\mu^{(2+2)\text{-}ll} &= \frac{25}{81}\Big[\,3A_1 m_{\pi^+}^{n_1} - A_3 m_{\pi^0}^{n_3} + A_4 m_\eta^{n_4} + A_5 m_{\eta'}^{n_5}\Big]\\
2a_\mu^{(2+2)\text{-}ls} &= \frac{5}{81}\Big[+6A_2 m_{K^+}^{n_2} - 4A_4 m_\eta^{n_4} + 2A_5 m_{\eta'}^{n_5}\Big]\\
a_\mu^{(4)\text{-}s} + a_\mu^{(3+1)\text{-}ss} + a_\mu^{(2+2)\text{-}ss} &= \frac{1}{81}\Big[+2A_2 m_{K^+}^{n_2} + 4A_4 m_\eta^{n_4} + A_5 m_{\eta'}^{n_5}\Big]
\end{aligned}$$

- Coefficients and powers laws (or logs) from chiral dependence of model or data
- Matching only expected to work well at low E (e.g., LD window)
- η-η' mixing not shown for simplicity

Detailed Example: $N_f = 2$ LO VP

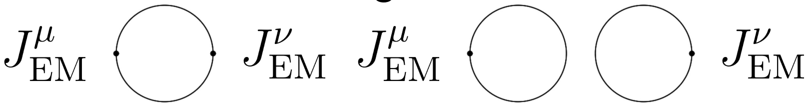
UV

IR

$C_q^{(2)}$  $D_{qq'}^{(1+1)}$ $\longleftrightarrow$ $\mathcal{I}_{\pi^+}, \mathcal{I}_{K^+} = \sum$ 

1. UV Diagrams & Current Correlators

Diagrams



$$C_q = Q_q^2 X_q \qquad\qquad D_{q_1 q_2} = Q_{q_1} Q_{q_2} \Delta_{q_1 q_2}$$

- X,Δ don't depend on QED charges, so are unaffected by QED isospin-breaking effects

Currents in quark- and isospin-bases

$$J^\mu_{\text{EM}} = Q_u J^\mu_u + Q_d J^\mu_d = (Q_u + Q_d) J^\mu_0 + (Q_u - Q_d) J^\mu_3$$

$$J^\mu_q = \bar{q}\gamma^\mu q, \qquad J^\mu_\alpha = \left(\bar{u}\ \ \bar{d}\right) \gamma^\mu T_\alpha \begin{pmatrix} u \\ d \end{pmatrix}$$

Correlators (each basis)

$$C_{\text{VP}} = \langle J^\mu_{\text{EM}}(x) J^\nu_{\text{EM}}(y) \rangle$$
$$\xrightarrow{m_u = m_d = m_l} Q_u^2 (X_l + \Delta_{ll}) + Q_d^2 (X_l + \Delta_{ll}) + 2 Q_u Q_d \Delta_{ll}$$

$$\begin{aligned} C_{\text{VP}} = {} & Q_u^2 (\langle J^\mu_0 J^\nu_0 \rangle + \langle J^\mu_3 J^\nu_3 \rangle + \langle J^\mu_0 J^\nu_3 \rangle + \langle J^\mu_3 J^\nu_0 \rangle) \\ & + Q_d^2 (\langle J^\mu_0 J^\nu_0 \rangle + \langle J^\mu_3 J^\nu_3 \rangle - \langle J^\mu_0 J^\nu_3 \rangle - \langle J^\mu_3 J^\nu_0 \rangle) \\ & + 2 Q_u Q_d (\langle J^\mu_0 J^\nu_0 \rangle - \langle J^\mu_3 J^\nu_3 \rangle) \end{aligned}$$

Monomial orthogonality

$$\Longrightarrow \begin{cases} \langle J^\mu_0 J^\nu_3 \rangle + \langle J^\mu_3 J^\nu_0 \rangle = 0 \\ X_l = 2 \langle J^\mu_3 J^\nu_3 \rangle \\ \Delta_{ll} = \langle J^\mu_0 J^\nu_0 \rangle - \langle J^\mu_3 J^\nu_3 \rangle \end{cases}$$

2. Connected Generating Functionals

Lagrangians (with sources j)

$$\begin{aligned} D^\mu &= \partial^\mu + ieB^\mu_\alpha[T^\alpha, \cdot] \equiv \partial^\mu + j^\mu_\alpha[T^\alpha, \cdot] \\ \mathcal{L}^{\mathrm{UV}} &= \bar{q}\not{D}q = \bar{q}\not{\partial}q + j^\alpha_\mu J^\mu_\alpha \\ \mathcal{L}^{\mathrm{IR}} &= \frac{F_0^2}{4}\mathrm{tr}\left[D_\mu U D^\mu U^\dagger\right] \\ &= \frac{1}{4}\mathrm{tr}\left[\partial_\mu\phi\partial^\mu\phi\right] + j^\alpha_\mu V^\mu_\alpha + j^\alpha_\mu j^\beta_\nu T^{\mu\nu}_{\alpha\beta} \end{aligned}$$

Noether Currents

$$(J_{\mathrm{Noether}}[j])^\mu_\alpha(x) = \begin{cases} J^\mu_\alpha(x), & \mathrm{UV} \\ V^\mu_\alpha(x) + j^\beta_\nu(x) T^{\mu\nu}_{\alpha\beta}(x), & \mathrm{IR} \end{cases}$$

- IR Noether current is a functional of the sources

Connected Generating Functional*

$$i(-i)^2 \frac{\delta}{\delta j^\alpha_\mu(x)} \frac{\delta W}{\delta j^\beta_\nu(y)}\bigg|_{j=0} = \begin{cases} \left\langle J^\mu_\alpha(x) J^\nu_\beta(y)\right\rangle_{\mathrm{conn}}, & \mathrm{UV} \\ \left\langle V^\mu_\alpha(x) V^\nu_\beta(y)\right\rangle_{\mathrm{conn}} & \\ \quad -2i\left\langle T^{\mu\nu}_{\alpha\beta}(x)\right\rangle_{\mathrm{conn}} \delta(x-y), & \mathrm{IR} \end{cases}$$

* “Connected” generating functional includes gluons in UV, so contains quark-line disconnected diagrams

UV Correlators (Reminder)

$$X_l = 2\langle J_3^\mu J_3^\nu \rangle$$
$$\Delta_{ll} = \langle J_0^\mu J_0^\nu \rangle - \langle J_3^\mu J_3^\nu \rangle$$

Connected Generating Functional (Reminder)

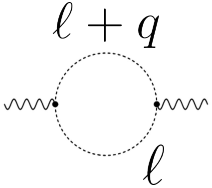
$$i(-i)^2 \frac{\delta}{\delta j_\mu^\alpha(x)} \frac{\delta W}{\delta j_\nu^\beta(y)}\bigg|_{j=0} = \begin{cases} \langle J_\alpha^\mu(x) J_\beta^\nu(y) \rangle_{\text{conn}}, & \text{UV} \\ \langle V_\alpha^\mu(x) V_\beta^\nu(y) \rangle_{\text{conn}} \\ \quad -2i \langle T_{\alpha\beta}^{\mu\nu}(x) \rangle_{\text{conn}} \delta(x-y), & \text{IR} \end{cases}$$

$$\mathcal{L}^{\text{IR}} \supset j_\mu^\alpha V_\alpha^\mu + j_\mu^\alpha j_\nu^\beta T_{\alpha\beta}^{\mu\nu}$$

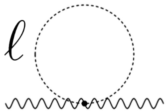
IR Diagrams

$$\langle V_0^\mu(x) V_0^\nu(y) \rangle, \langle T_{00}^{\mu\nu}(x) \rangle = O(\phi^3)$$

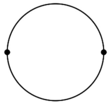
$$\begin{aligned} \langle V_3^\mu(x) V_3^\nu(y) \rangle &\to -\ell^\mu(\ell^\nu + q^\nu)\,\pi^+(\ell)\pi^-(\ell+q) + \ldots \\ &\to -2\ell^\mu(2\ell^\nu + q^\nu)\,S_\pi(\ell) S_\pi(\ell+q) \end{aligned} \quad \longrightarrow$$



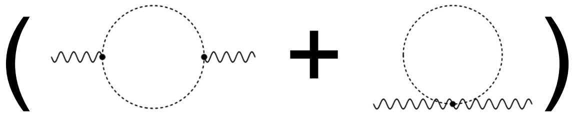
$$\begin{aligned} -2i \langle T_{33}^{\mu\nu}(x) \rangle \delta(x-y) &\to 2ig^{\mu\nu} \pi^+(\ell)\pi^-(\ell) \\ &\to 2ig^{\mu\nu} S_\pi(\ell) \end{aligned} \quad \longrightarrow$$

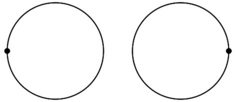


4. Matching Relations

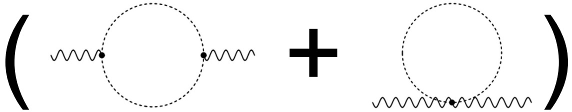


$$= (Q_u^2 + Q_d^2) X_l = (Q_u^2 + Q_d^2)\,(2\langle J_3 J_3 \rangle) \to \frac{10}{9}$$





$$= (Q_u + Q_d)^2 \Delta_l \to -\frac{1}{9}$$



$$C_l^{(2)} = \frac{10}{9} \mathcal{I}_{\pi^+}^{\mathrm{VP}}$$

$$D_{ll}^{(1+1)} = -\frac{1}{9} \mathcal{I}_{\pi^+}^{\mathrm{VP}}$$

- Reproduced well-known result (ratio of conn to disc at low E)
- Also agrees with PQ approach