
Precision Determinations of the Fine-structure Constant

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JOHANNES GUTENBERG
UNIVERSITÄT MAINZ





Organisers: Klaus Blaum, Martin Hoferichter, Makiko Nio, H.W.

Topics covered

Experimental measurements of α and a_e

S. Guellati-Khelifa, H. Müller, J. Roth, J. Schelfhout, G. Gabrielse

Muonium spectroscopy and implications for α and $g - 2$

E. Thorpe-Woods, P. Strasser, S. Kamioka

Precision tests of QED and nuclear theory

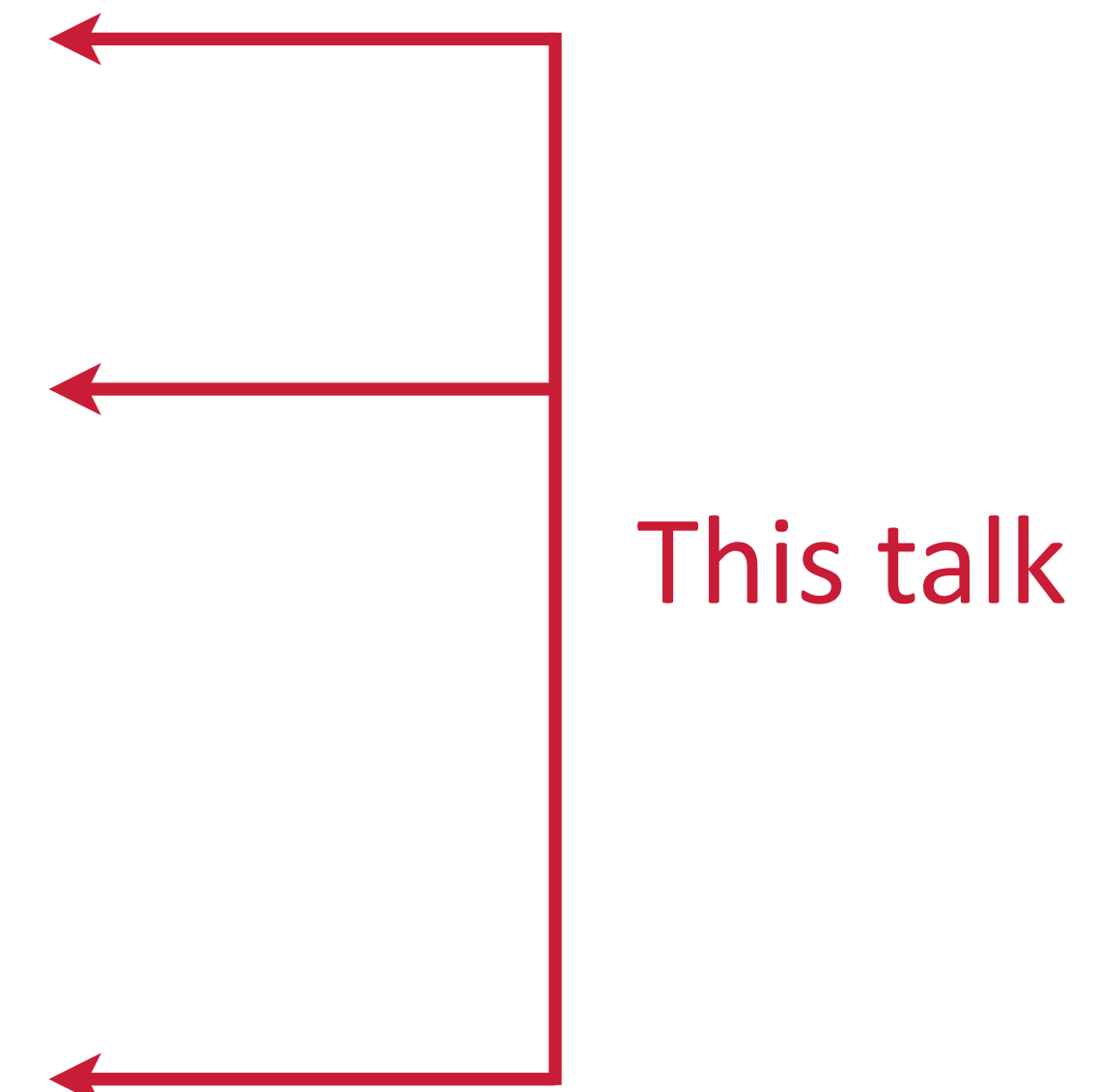
S. Sturm, A. Mooser, N. Oreshkina, R. Pohl, B. Sikora

Status & prospects for QED contribution to a_e

S. Laporta, S. Volkov

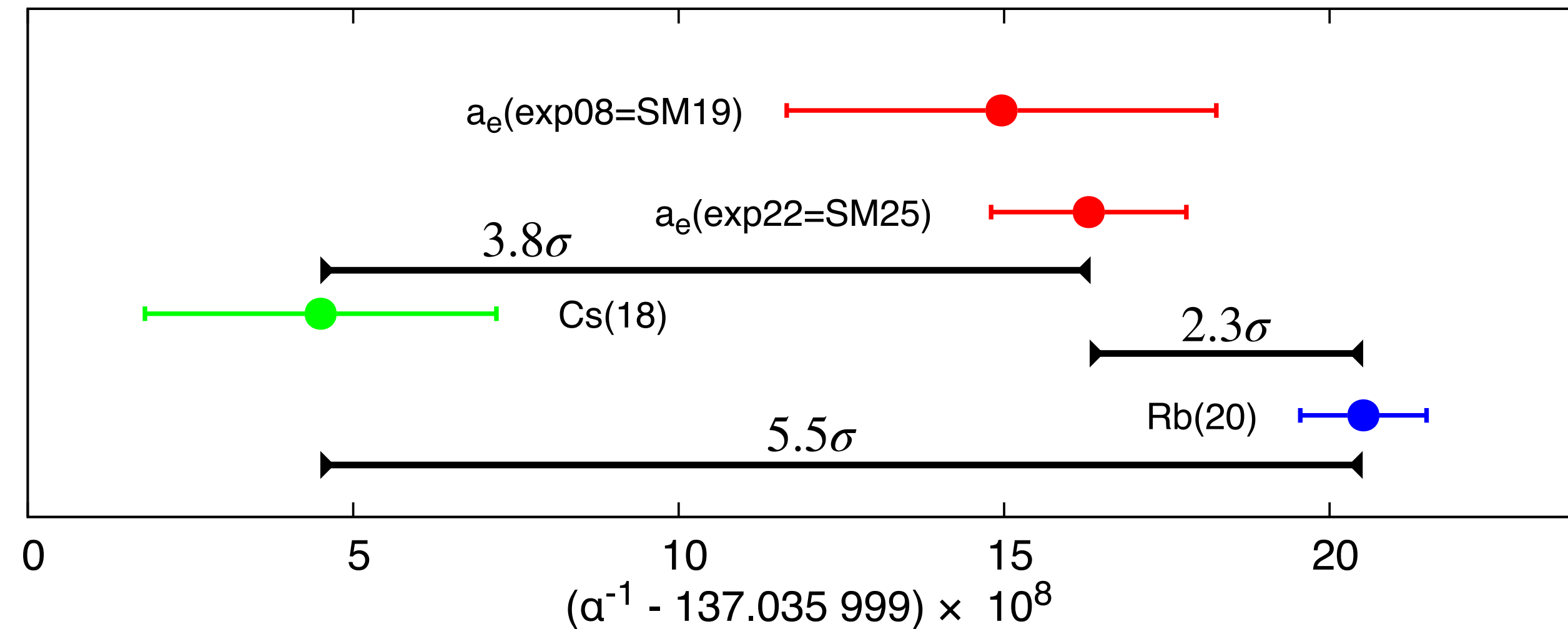
Hadronic contributions to a_e and the running of α

A. Wright, D. Giusti, W. Wang



<https://indico.mitp.uni-mainz.de/event/417/overview>

Fine-structure Constant: The State of Affairs



Conflicting measurements from atomic recoil interferometry of **Cesium**, **Rubidium**, as well as from QED contribution to a_e

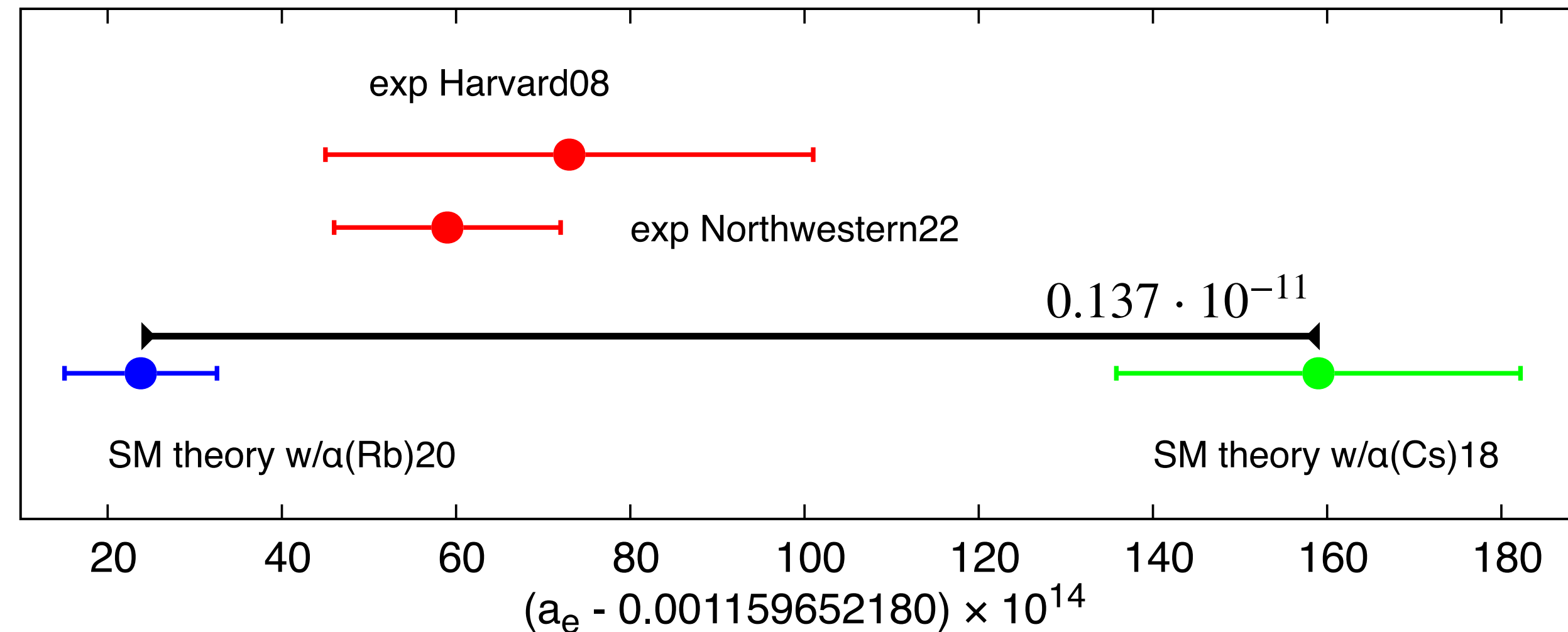
$$\alpha^{-1}(\text{Cs}) = 137.035\,999\,045(27) \quad [200 \text{ ppt}]$$

$$\alpha^{-1}(\text{Rb}) = 137.035\,999\,2052(97) \quad [71 \text{ ppt}]$$

$$\alpha^{-1}(a_e) = 137.035\,999\,163(15) \quad [110 \text{ ppt}]$$

[WP 25: Aliberti et al., [arXiv:2505.21476](https://arxiv.org/abs/2505.21476)]

Fine-structure Constant: The State of Affairs



Difference between estimates of a_μ^{QED} as large as neglected 12th order (6 loop) contribution

$$\Rightarrow a_\mu^{\text{QED}} = 116\,584\,718.8(2) \cdot 10^{-11}$$

- Resolve tension $\longleftrightarrow$ understand systematics; improve precision further
- Additional / complementary measurements with competitive / better precision
- Implications for SM tests based on a_e

[WP 25: Aliberti et al., [arXiv:2505.21476](https://arxiv.org/abs/2505.21476)]

The electron anomalous magnetic moment

[Gerald Gabrielse]

Measurement of a_e in a “one-electron cyclotron”
(Penning trap in a cryogenic system)

Single electron in magnetic field, cooled to its lowest quantum states

$$E = m_s h \nu_s + (n + \frac{1}{2}) h \nu_c \quad \nu_c = \frac{eB}{2\pi m}, \quad \nu_s = \frac{g}{2} \nu_c, \quad \nu_a = \nu_s - \nu_c$$

$$\Rightarrow a_e = \frac{g}{2} - 1 = \frac{\nu_a}{\nu_c} \quad (B\text{-field drops out in frequency ratios})$$

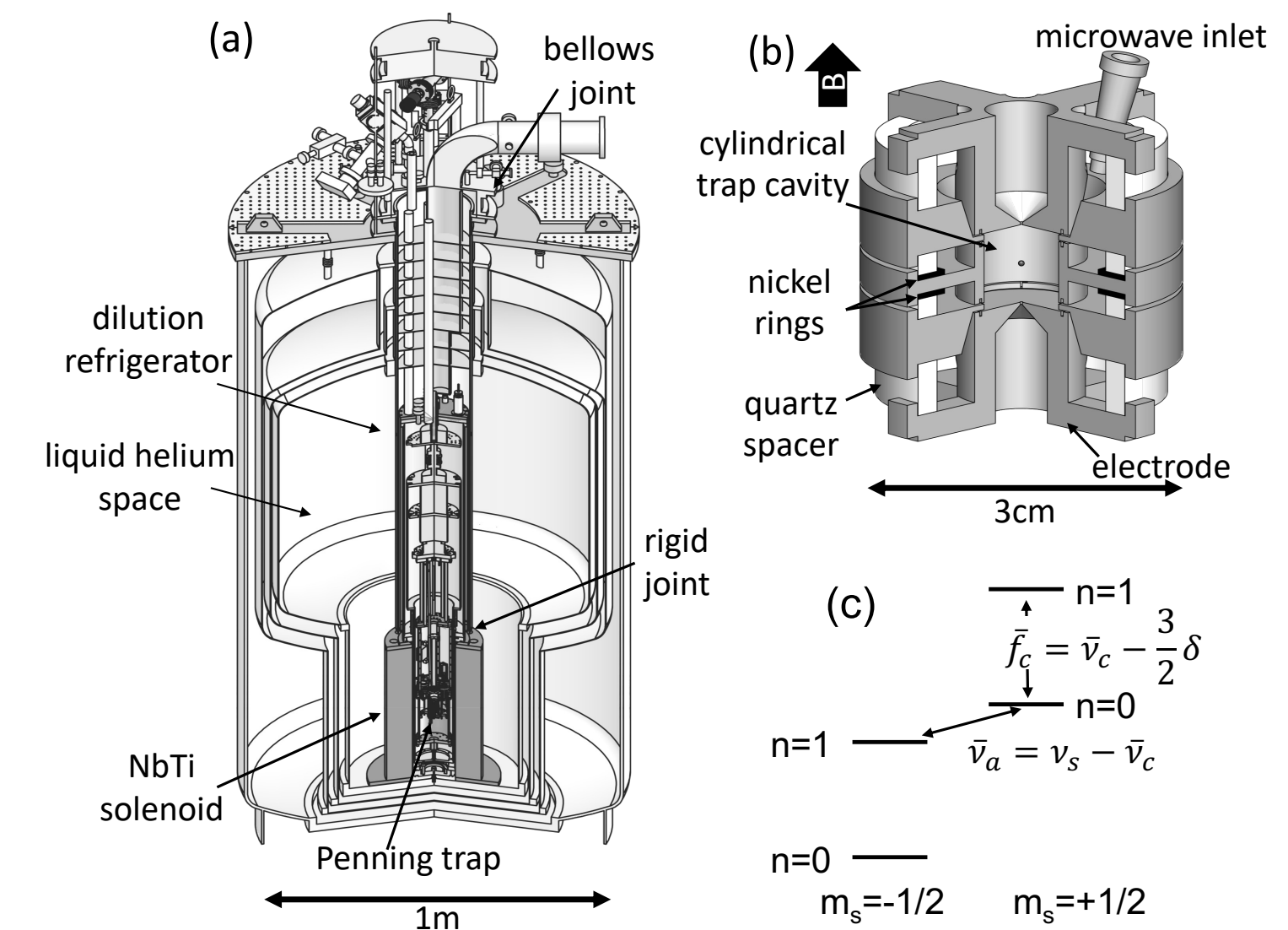
$$= 0.001\,159\,652\,180\,59(13) \quad [0.13 \text{ ppt}]$$

Convert into estimate of α :

$$a_e = \sum_{n=1}^{\infty} \left(\frac{\alpha}{\pi}\right)^n A_1^{(2n)} + a_e^{\mu\tau} + a_e^{\text{weak}} + a_e^{\text{strong}} \quad \Rightarrow \quad \alpha^{-1}(a_e) = 137.035\,999\,163(15) \quad [110 \text{ ppt}]$$

Work towards improvement by another order of magnitude

[Fan et al., Phys Rev Lett 130 (2023) 071801]



Source	Uncertainty $\times 10^{13}$
statistical	0.29
cyclotron broadening	0.94
cavity correction	0.90
nuclear paramagnetism	0.12
anomaly power shift	0.10
magnetic field drift	0.09
total	1.3

α from atomic recoil interferometry

Master formula:

$$hc R_\infty = \frac{1}{2} m_e \alpha^2 c^2 \quad \Rightarrow \quad \alpha^2 = \frac{2 R_\infty}{c} \frac{h}{m_e} = \frac{2 R_\infty}{c} \frac{A_r(X)}{A_r(e)} \frac{h}{m(X)}$$

Relative atomic mass of atom X : $A_r(X) = \frac{m(X)}{m_u}$, $m_u = \frac{1}{12} m(^{12}\text{C})$

Measure $\frac{h}{m(X)}$, $X = \text{Cs, Rb, ...}$ via atomic recoil interferometry

Quantity	u_r (rel. uncertainty)
R_∞	$1.1 \cdot 10^{-12}$
$A_r(e)$	$1.8 \cdot 10^{-11}$
$A_r(^{87}\text{Rb})$	$7.0 \cdot 10^{-11}$
$A_r(^{133}\text{Cs})$	$6.5 \cdot 10^{-11}$

Tension between $\alpha(\text{Rb})$ and $\alpha(\text{Cs})$

$\Leftrightarrow$ discrepant determinations of atomic mass unit

$$m_u \equiv \frac{m(X)}{A_r(X)} = \begin{cases} 1.660\,539\,069\,61(28) \cdot 10^{-27} \text{ kg (Rb)} \\ 1.660\,539\,065\,73(68) \cdot 10^{-27} \text{ kg (Cs)} \end{cases}$$

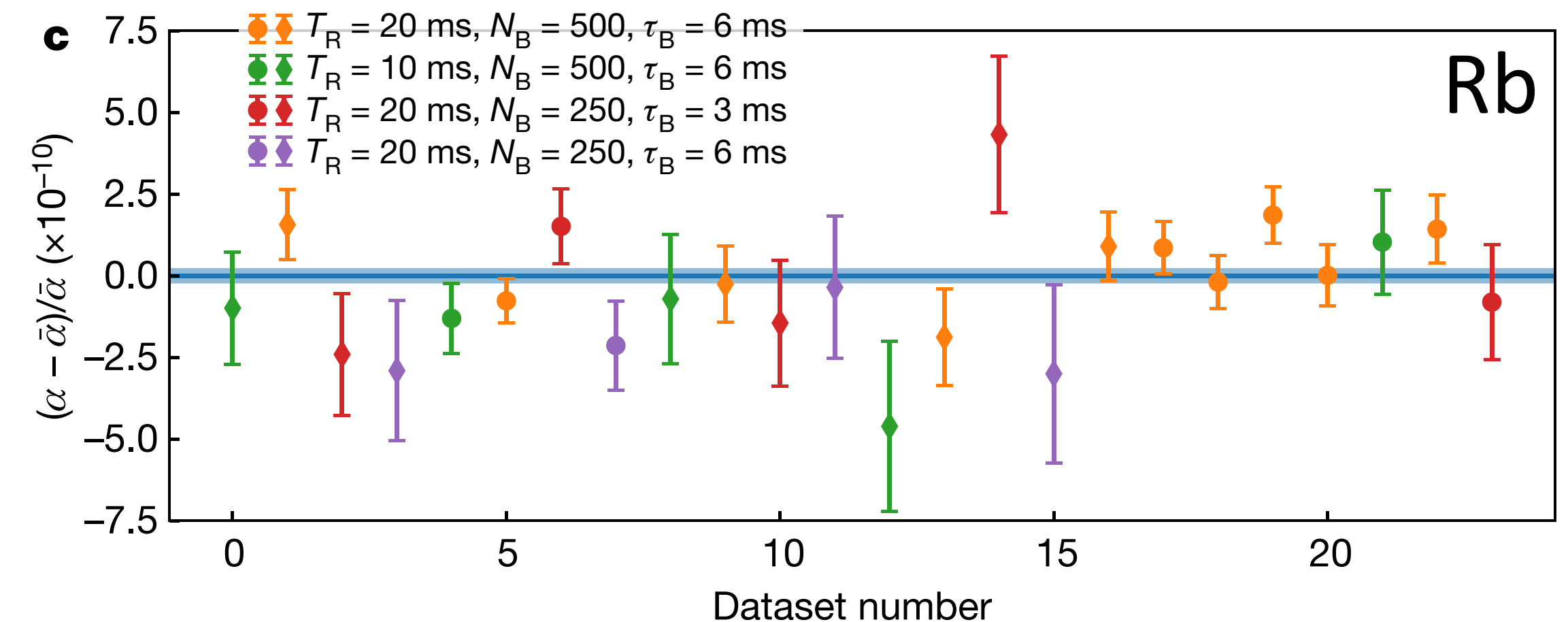
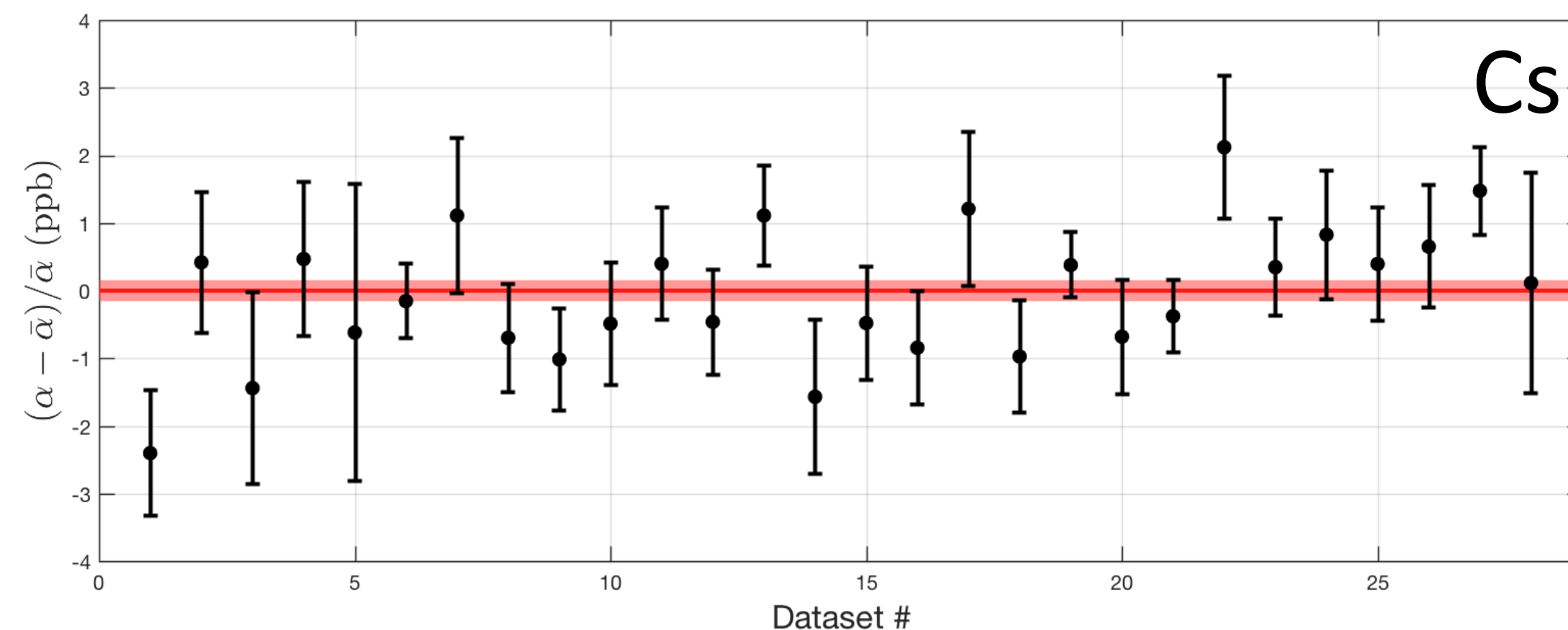
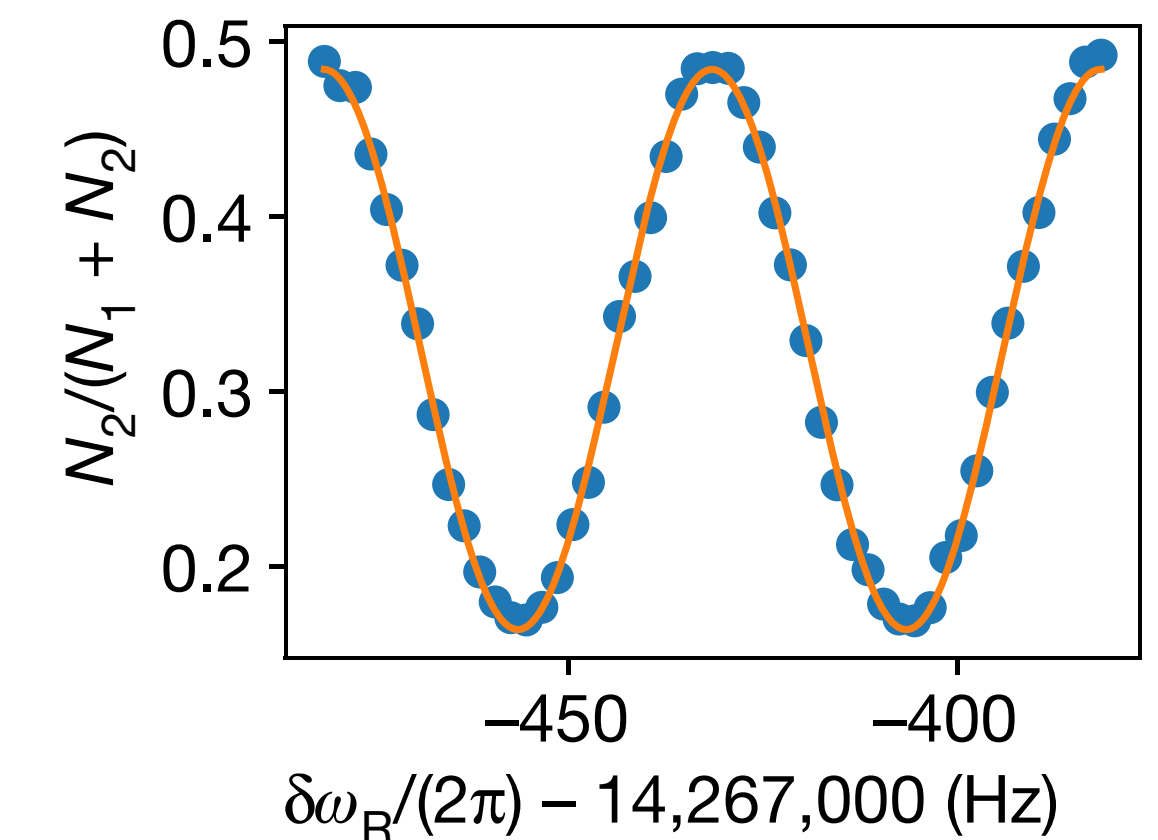
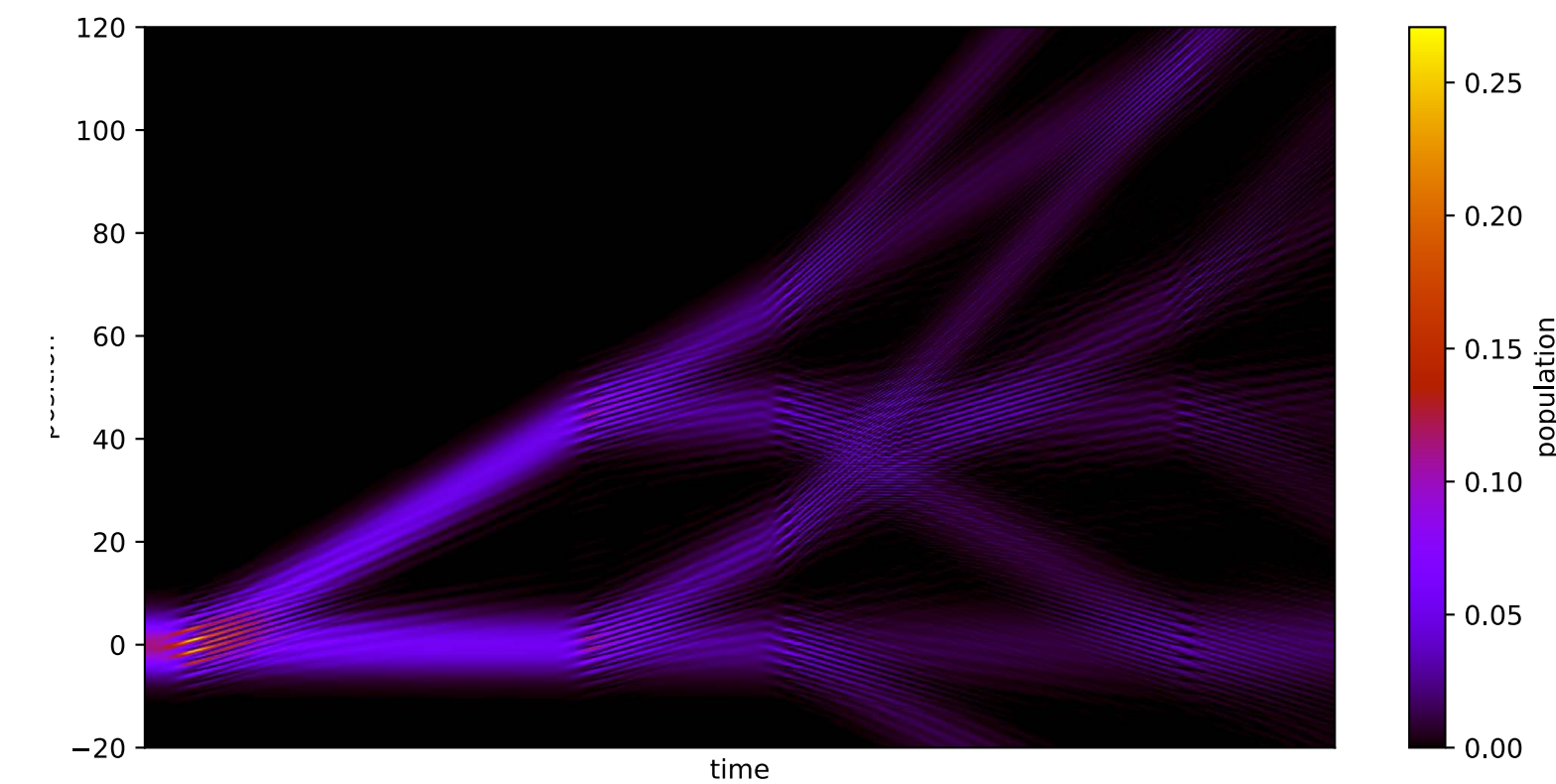
$\Rightarrow$ SM test limited by knowledge of $m(^{12}\text{C})$

Atomic recoil interferometry

[Saïda Guellati-Khelifa, Holger Mueller, Jack Roth]

Procedure

- Coherent transfer of a precise number of photons to atoms; measure phase shift due to change in atom's velocity, using a matter-wave interferometer
- Split and recombine atomic light packets via sequence of light pulses
- Sinusoidal variation of the accumulated phase difference between atomic populations along the two paths
- Phase shift proportional to $\hbar/m(X)$

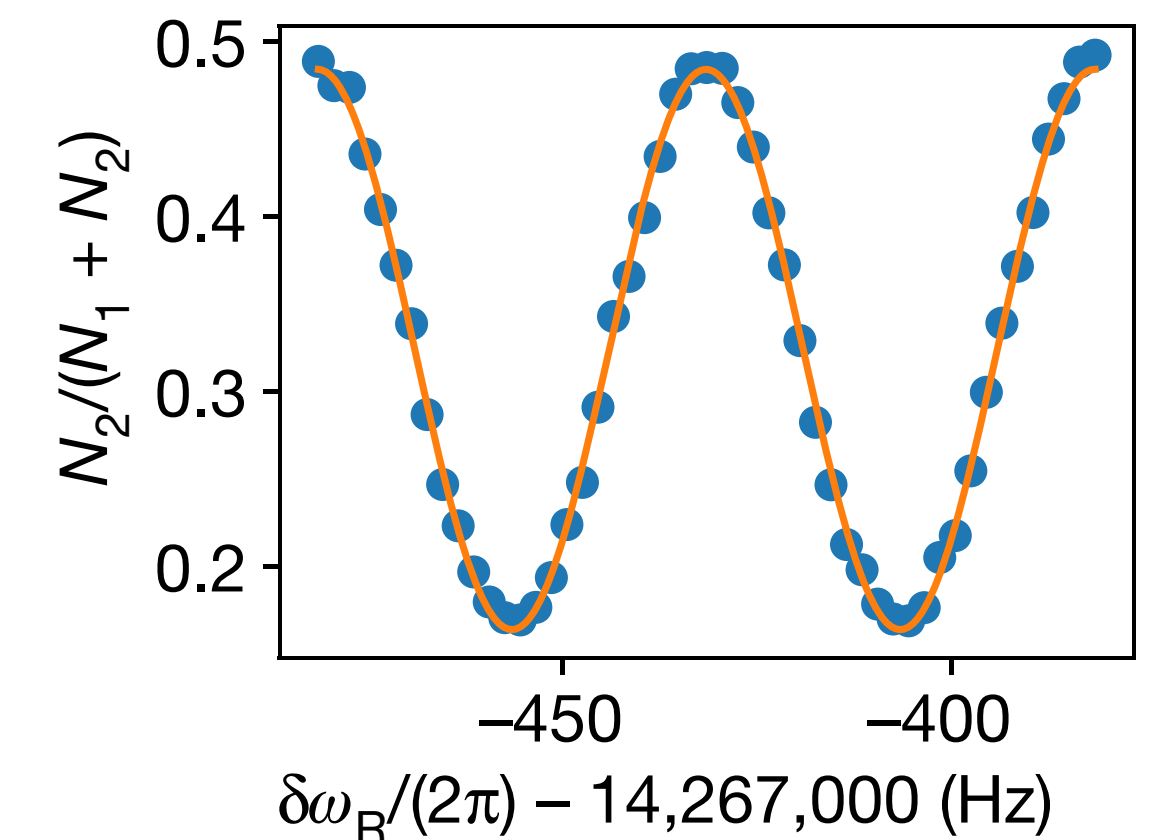
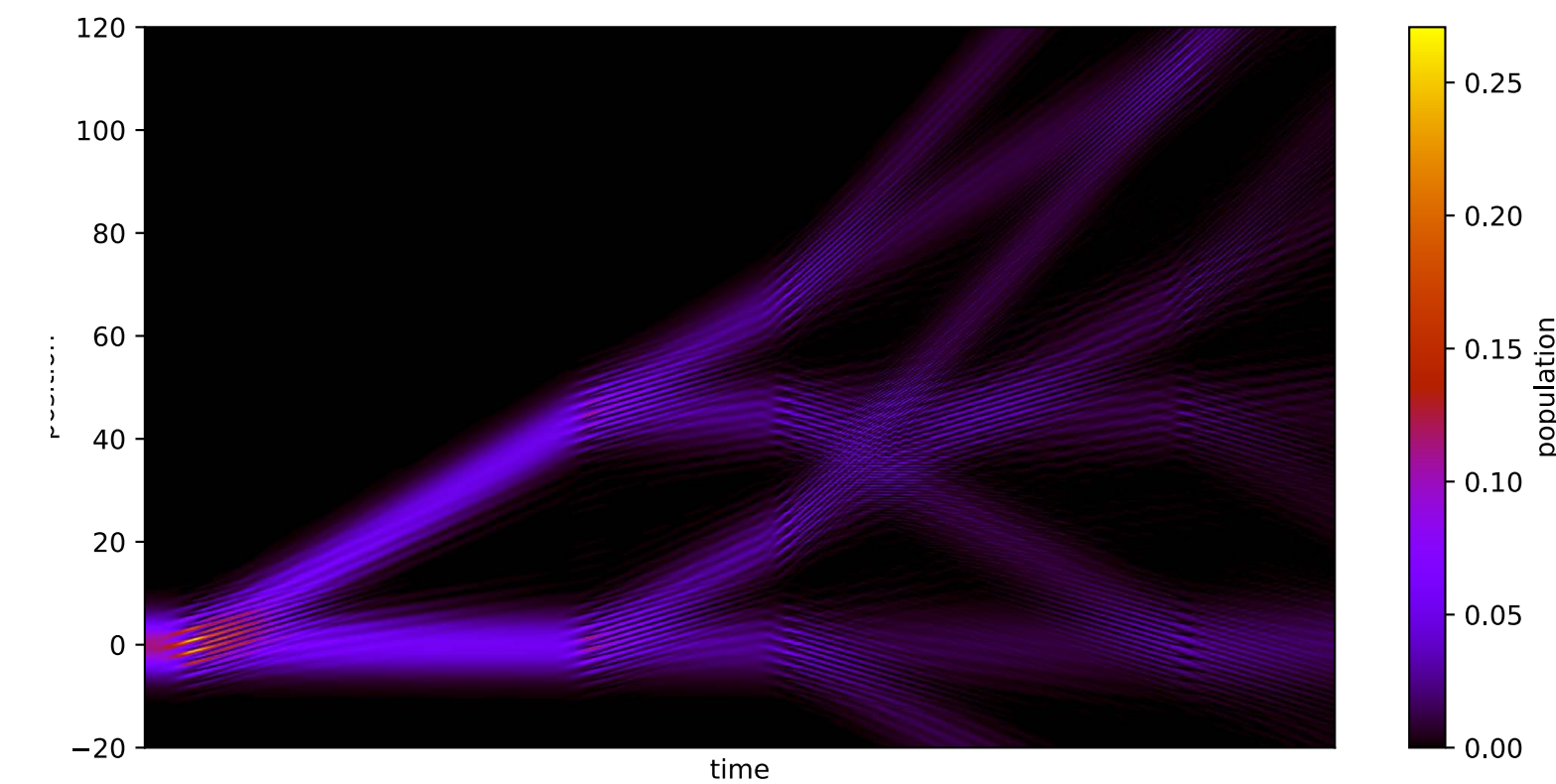


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Berkeley:

$$\hbar/m(\text{Cs}) = 3.002\,369\,4721(12) \cdot 10^{-9} \text{ m}^2/\text{s}$$

[Parker et al., Science 360 (2018) 191]

Paris (LKB):

$$\hbar/m(\text{Rb}) = 4.591\,359\,258\,90(65) \cdot 10^{-9} \text{ m}^2/\text{s}$$

[Morel et al., Nature 588 (2020) 61]

Atomic recoil interferometry — Systematics

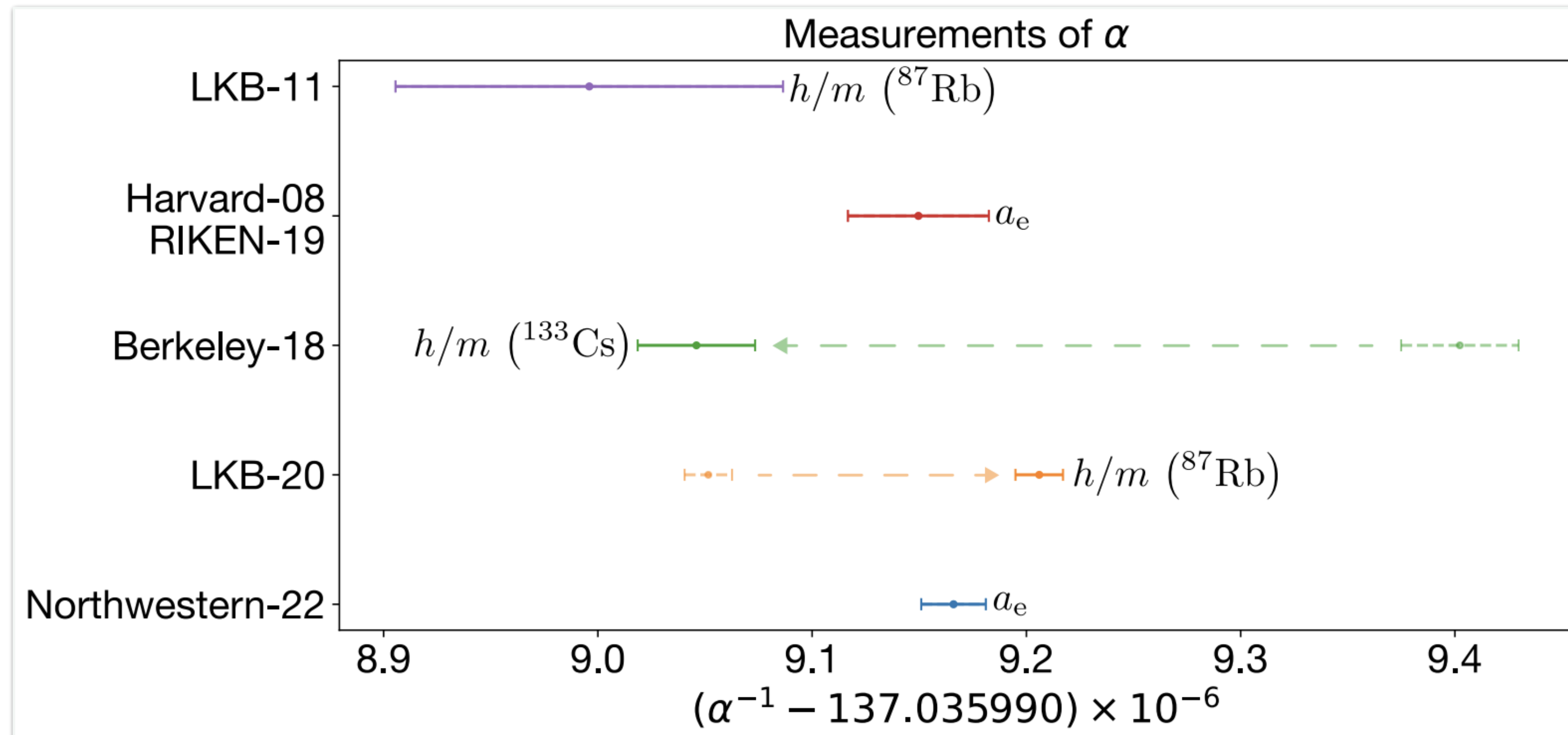
Paris (LKB)

Source	Correction [10^{-11}]	Relative uncertainty [10^{-11}]
Gravity gradient	-0.6	0.1
Alignment of the beams	0.5	0.5
Coriolis acceleration		1.2
Frequencies of the lasers		0.3
Wave front curvature	0.6	0.3
Wave front distortion	3.9	1.9
Gouy phase	108.2	5.4
Residual Raman phase shift	2.3	2.3
Index of refraction	0	< 0.1
Internal interaction	0	< 0.1
Light shift (two-photon transition)	-11.0	2.3
Second order Zeeman effect		0.1
Phase shifts in Raman phase lock loop	-39.8	0.6
Global systematic effects	64.2	6.8
Statistical uncertainty		2.4
Relative mass of $^{87}\text{Rb}^{16}$: 86.909 180 531 0(60)		3.5
Relative mass of the electron 14 : 5.485 799 090 65(16) $\cdot 10^{-4}$		1.5
Rydberg constant 14 : 10 973 731.568 160(21) m^{-1}		0.1
Total: $\alpha^{-1} = 137.035\,999\,206(11)$		8.1

Berkeley

Effect	Section	$\delta\alpha/\alpha$ (ppb)
<i>This study</i>		
Laser frequency	1	-0.24 ± 0.03
Acceleration gradient	4A	-1.79 ± 0.02
Gouy phase	3	-2.60 ± 0.03
Beam alignment	5	0.05 ± 0.03
Bloch oscillation light shift	6	0 ± 0.002
Density shift	7	0 ± 0.003
Index of refraction	8	0 ± 0.03
Speckle phase shift	4B	0 ± 0.04
Sagnac effect	9	0 ± 0.001
Modulation frequency wave number	10	0 ± 0.001
Thermal motion of atoms	11	0 ± 0.08
Non-Gaussian waveform	13	0 ± 0.03
Parasitic interferometers	14	0 ± 0.03
Total systematic error	All previous	-4.58 ± 0.12
Statistical error	N/A	± 0.16
<i>Other studies</i>		
Electron mass (16)	N/A	± 0.02
Cesium mass (6, 15)	N/A	± 0.03
Rydberg constant (6)	N/A	± 0.003
<i>Combined result</i>		
Total uncertainty in α	N/A	± 0.20

Atomic recoil interferometry — Systematics



Beam-related systematics are large

Rb and Cs measurements — Status and next steps

Common problem:

Spatial distribution of wave vectors $\vec{k}$ along beam front — **Gouy phase**

- Measured beam profile fed into MC simulation (Berkeley)
- Numerical simulation of recoil velocity measurement (LKB)
- 2D BEC array & imaging techniques to probe different regions of transverse laser beam profile (LKB)

Other systematics:

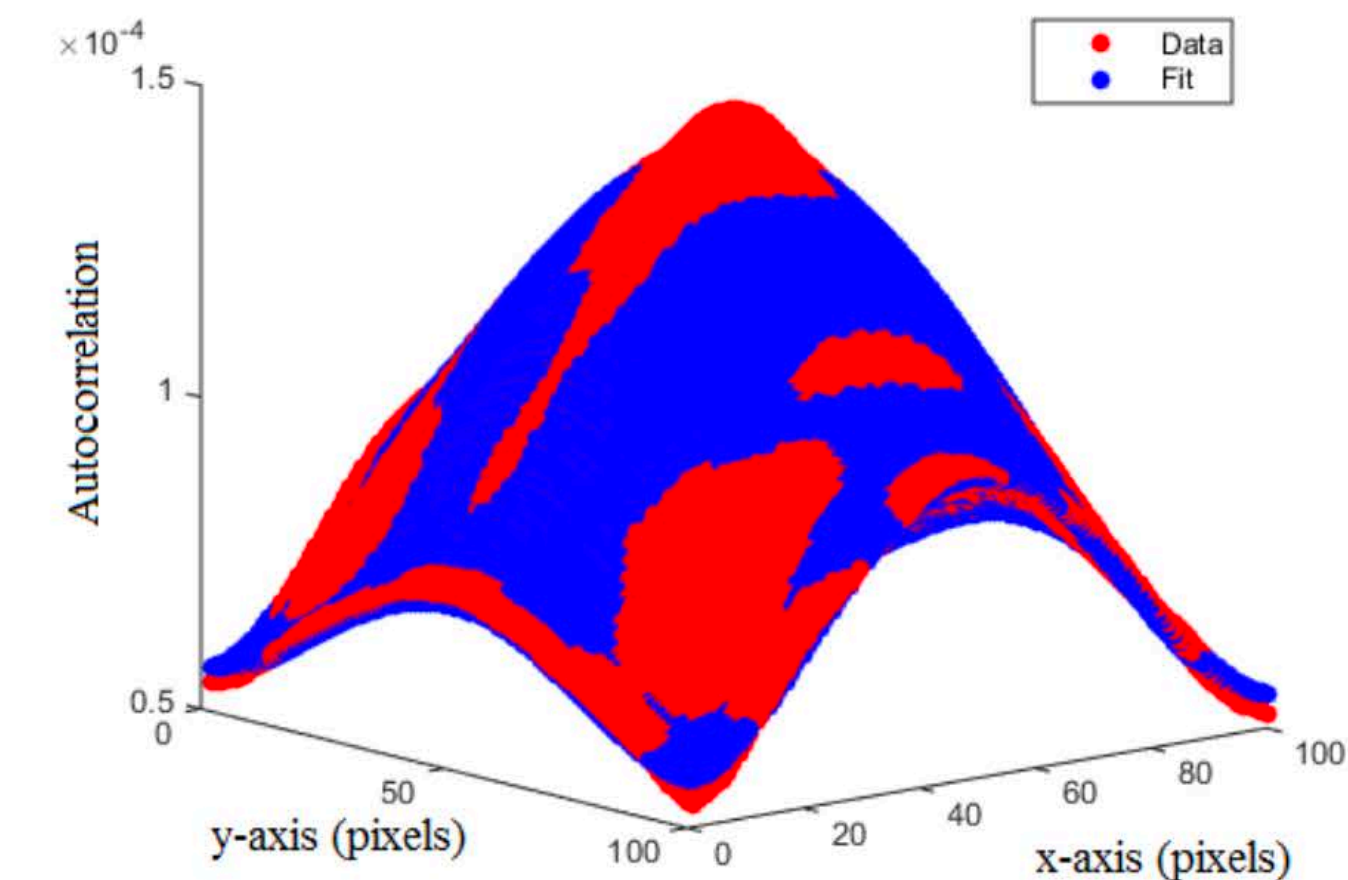
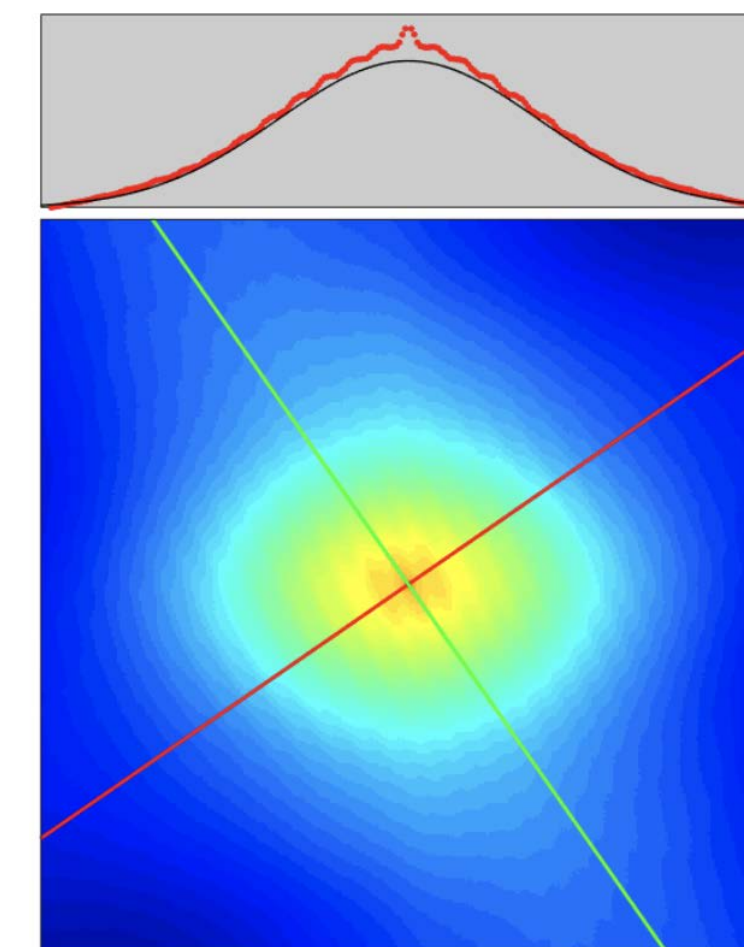
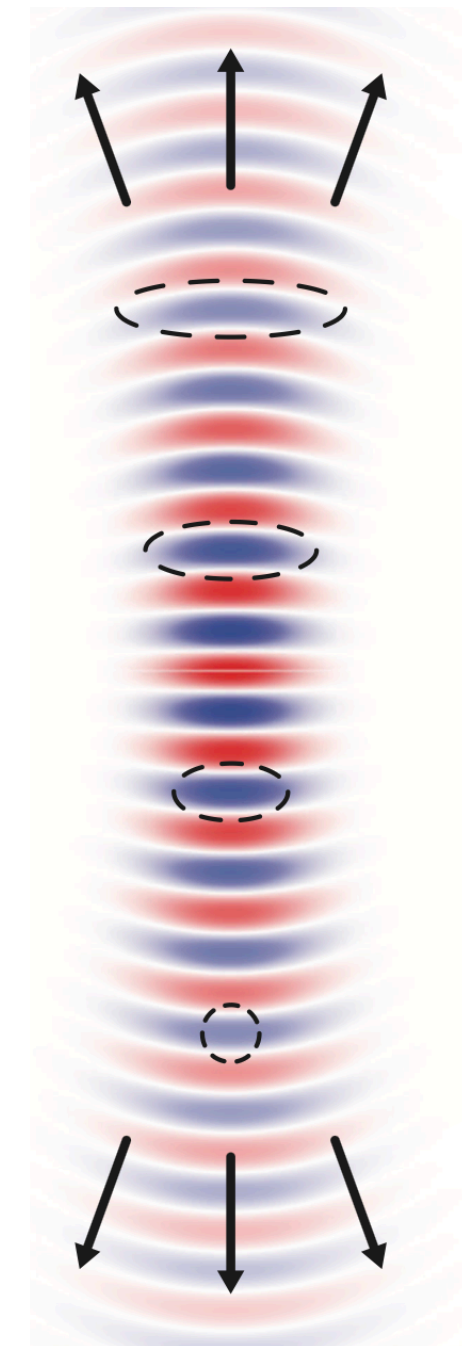
Small intensity ripples (speckles) cause shift in $\vec{k}$

- suppressed by apodising filter (Berkeley)

Mechanical vibrations

New experimental hardware

Develop blinding scheme for data taking



Intermediate-scale Clock-atom interferometry

[Jan Schelfhout]

Exploit long coherence times of single-photon optical clock transitions

$$\text{Sr}(5s^2)^1S_0 - \text{Sr}(5s5p)^3P_1 \sim 150 \text{ s} \quad \text{Yb}(6s^2)^1S_0 - \text{Yb}(6s6p)^3P_1 \sim 20 \text{ s}$$

Use intermediate-scale ($\sim 10 \text{ m}$) gravitational wave detector setups with clock atoms

Larger laser beams — reduced curvature in laser wavefronts

$$u_r(\alpha) = \frac{\Delta\alpha}{\alpha} \approx \left\{ \left(\frac{\Delta R_\infty}{R_\infty} \right)^2 + \left(\frac{\Delta A_r(e)}{A_r(e)} \right)^2 + \left(\frac{\Delta A_r(\text{Sr})}{A_r(\text{Sr})} \right)^2 + \left(\frac{\Delta m(\text{Sr})}{m(\text{Sr})} \right)^2 \right\}^{1/2}$$

Relative uncertainties of absolute mass measurements

$$u_r(m(^{87}\text{Sr})) = 2.2 \cdot 10^{-12}, \quad u_r(m(^{171}\text{Yb})) = 3.1 \cdot 10^{-12}$$

$$u_r(m(^{87}\text{Rb})) = 1.7 \cdot 10^{-10}$$

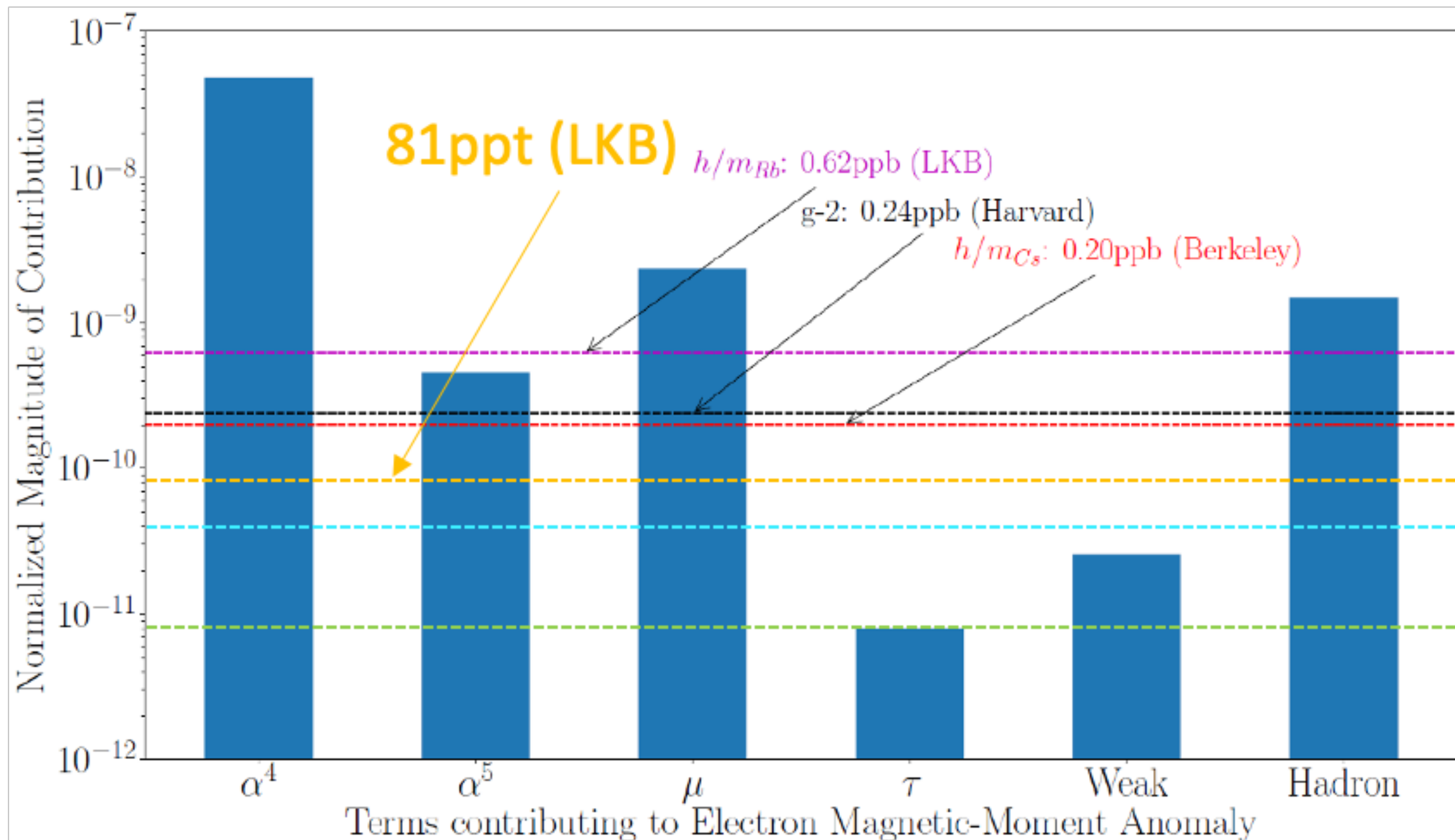
→ $u_r(\alpha)$ limited by **relative** mass measurement

Quantity	u_r (rel. uncertainty)
R_∞	$1.1 \cdot 10^{-12}$
$A_r(e)$	$1.8 \cdot 10^{-11}$
$A_r(^{87}\text{Rb})$	$7.0 \cdot 10^{-11}$
$A_r(^{87}\text{Sr})$	$7.0 \cdot 10^{-11}$
$A_r(^{171}\text{Yb})$	$8.2 \cdot 10^{-11}$

Intermediate-scale Clock-atom interferometry

[Jan Schelfhout]

Sensitivity of Standard Model test based on a_e



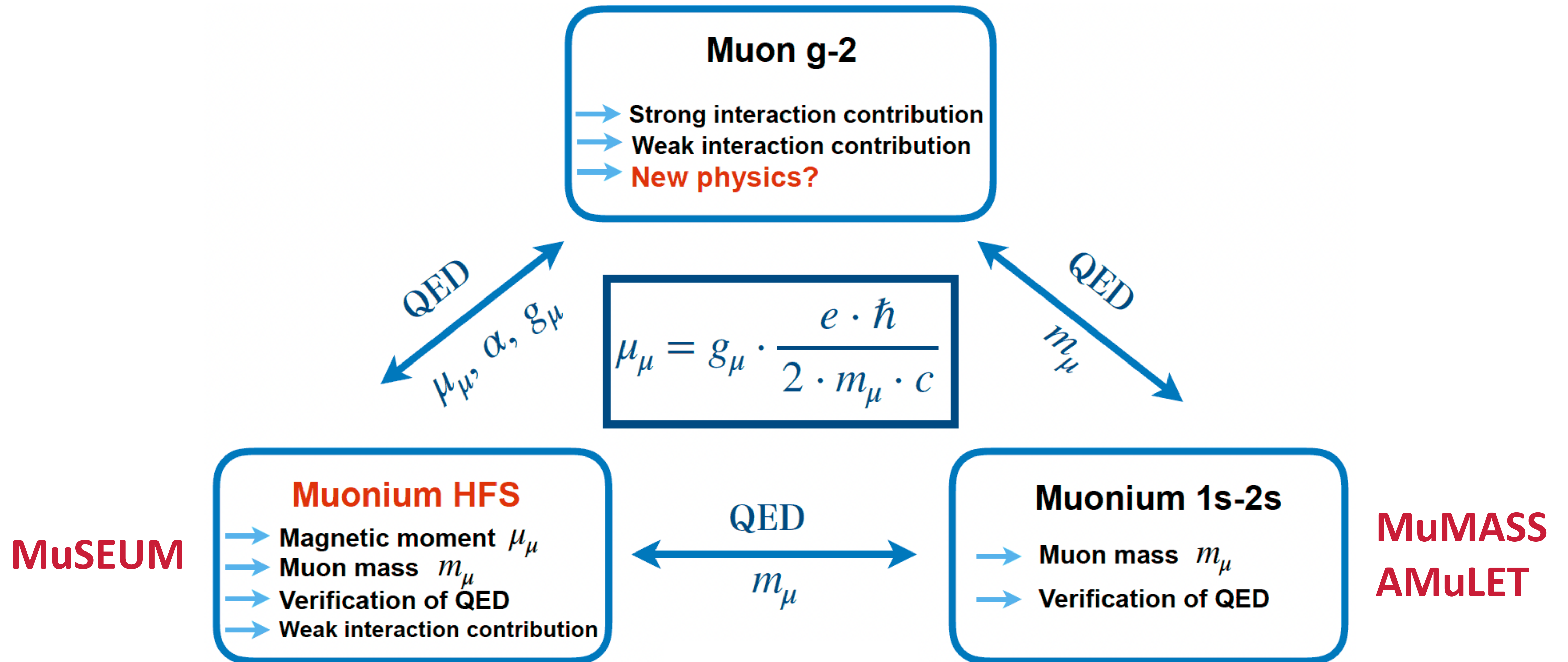
Limit due to current relative masses

Hypothetical factor-10 improvement

→ Sr-based measurement will improve $u_r(\alpha)$ by factor two

Muonium spectroscopy

[Edward Thorpe-Wood (MuMASS@PSI), Patrick Strasser & Shusei Kamioka (MuSEUM, AMuLET @J-PARC)]



[Slide from Klaus Jungmann]

Muonium spectroscopy

Muonium: μ^+e^- bound state

Purely leptonic system: no nuclear structure effects,
but large-ish experimental uncertainties

Ordinary (muonic) atoms: high experimental precision, but theoretically challenging

Muonium HFS governed by

Fine-structure constant α ; Rydberg constant R_∞

Electron-muon mass ratio m_e/m_μ ; Muon magnetic moment μ_μ

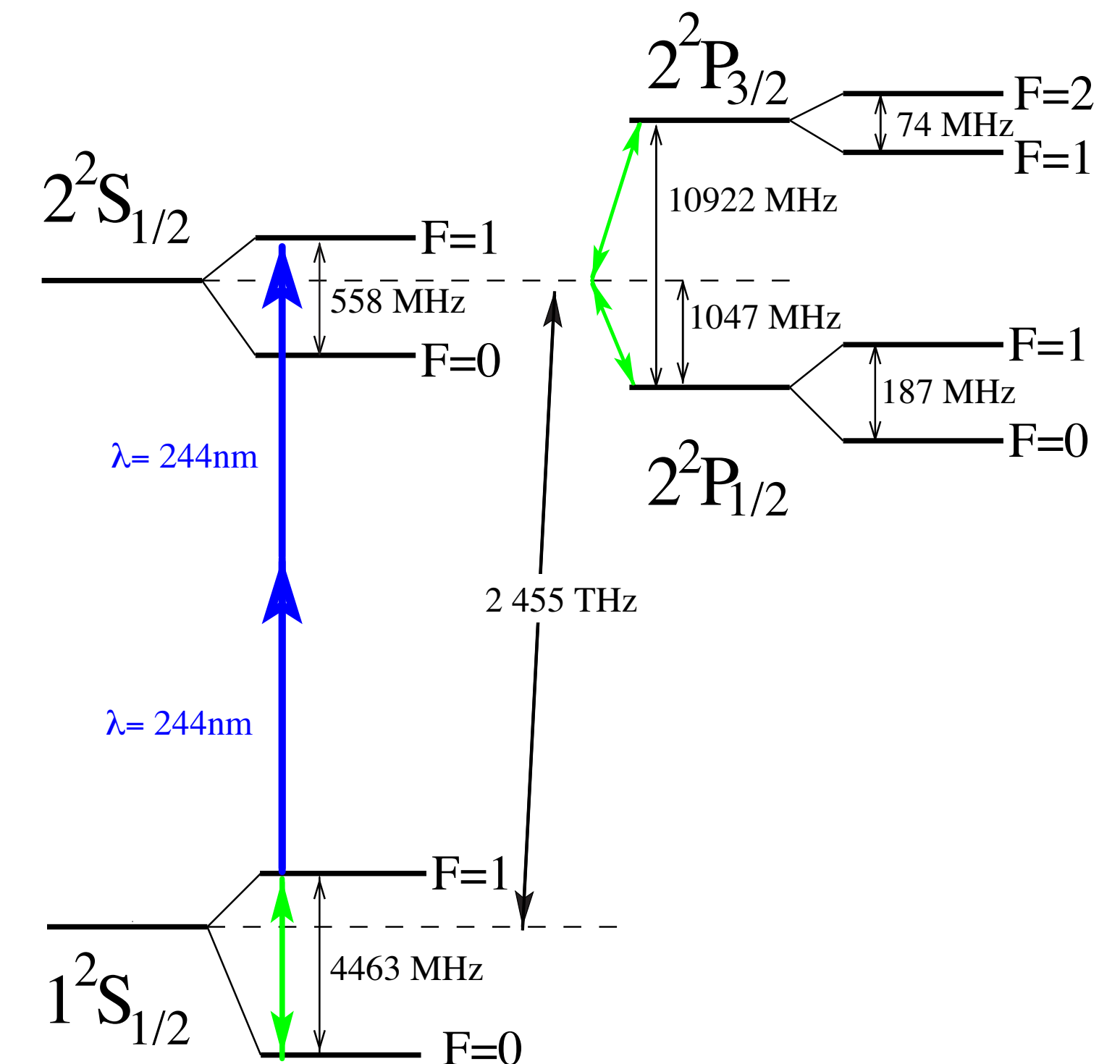
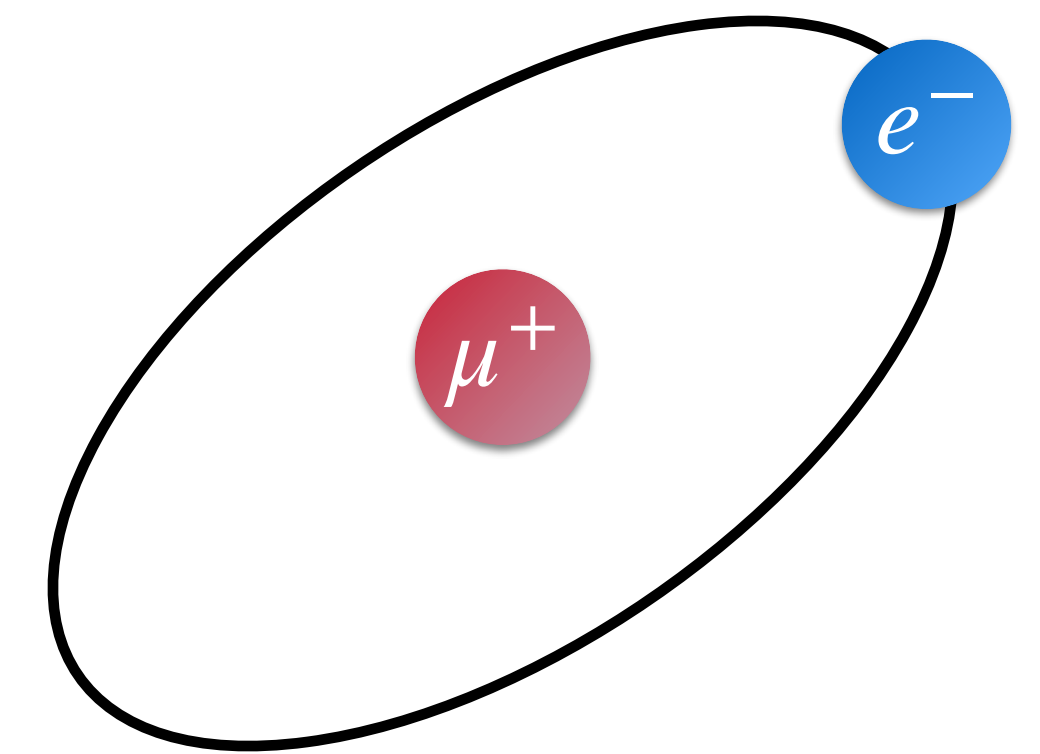
$$\nu_{1S-2S} = \frac{3}{4} \frac{R_\infty c}{(1 + m_e/m_\mu)} [1 + \delta_{1S-2S}]$$

$$\nu_{\text{HFS}} = \frac{16}{3} (1 + a_\mu) \frac{m_e}{m_\mu} \frac{R_\infty c \alpha^2}{(1 + m_e/m_\mu)^3} [1 + \delta_{\text{HFS}}]$$

→ determine m_e/m_μ independently from quantities that enter a_μ

→ Lyman- α line between 1S–2S states in muonium.

[Delaunay, Ohayon & Soreq, PRL 127 (2021) 25, 251801]



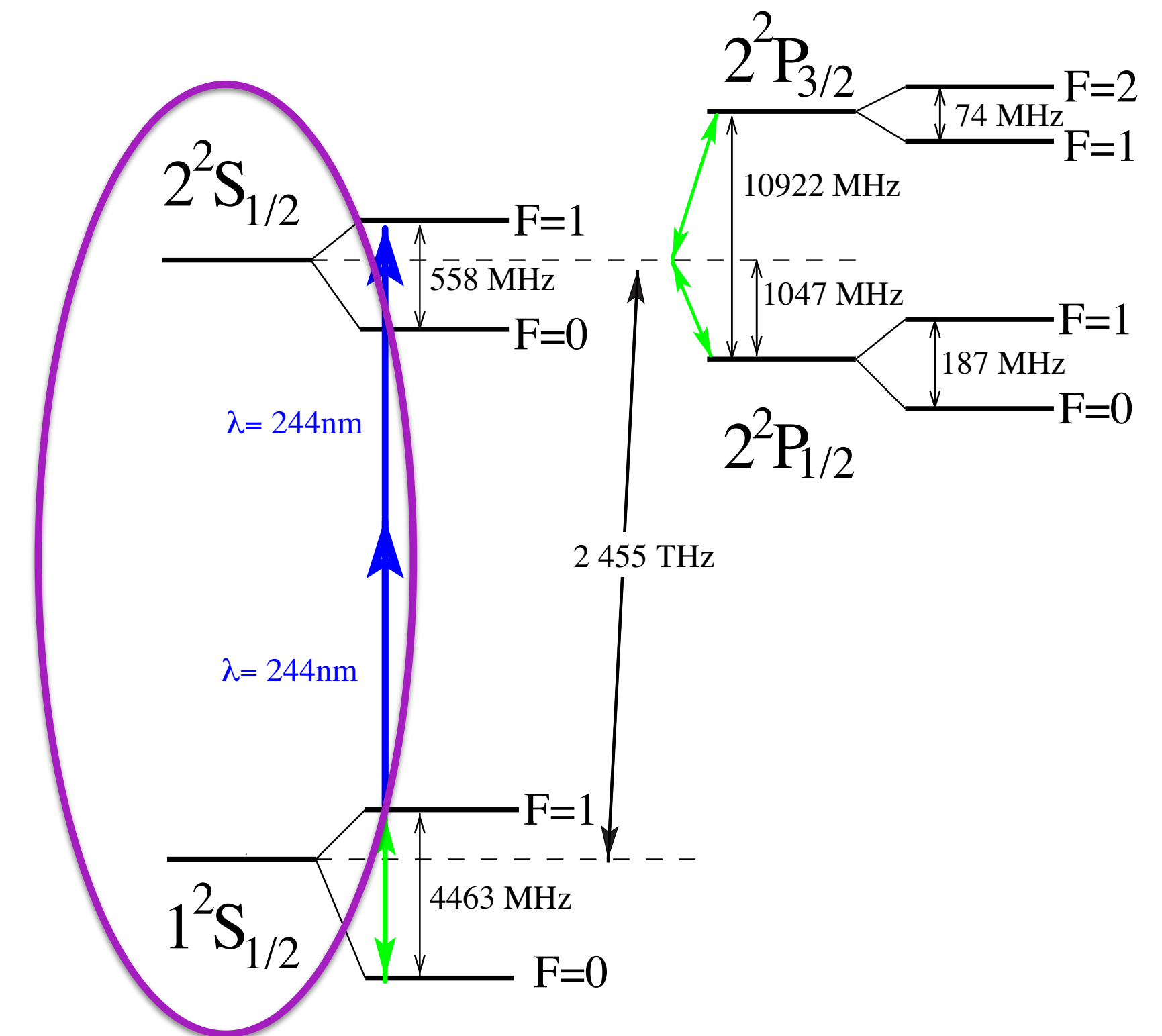
Mu-MASS, AMuLET and MuSEUM: α from HFS in muonium

Current experimental sensitivity of α limited by m_e/m_μ

$$\nu_{\text{HFS}} = \frac{16}{3} (1 + a_\mu) \frac{m_e}{m_\mu} \frac{R_\infty c \alpha^2}{(1 + m_e/m_\mu)^3} [1 + \delta_{\text{HFS}}] \quad \sigma_{\text{HFS}} = \sqrt{\frac{\delta_{\text{HFS}}^2}{85^2} + \frac{\sigma_{\text{exp}}^2}{51^2} + \frac{\sigma_{e\mu}^2}{511^2}} = 520 \text{ Hz}$$

Mu-MASS@PSI, AMuLET@J-PARC

- Reduce rel. uncertainty $u_r(m_e/m_\mu)$ from 120 ppb to 1 ppb
- Mu-MASS: measure 1S–2S transition with CW laser at LEM beam line to compensate low transition efficiency
- Assume that the HFS will be measured with $\sigma_{\text{exp}} = 5 - 8.5 \text{ Hz}$ and that δ_{HFS} will be improved to 5 – 10 Hz:



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- Assume that the HFS will be measured with $\sigma_{\text{exp}} = 5 - 8.5 \text{ Hz}$ and that δ_{HFS} will be improved to 5 – 10 Hz:

$$\nu_{\text{HFS}} \propto \alpha^4 \Rightarrow u_r(\alpha) = 4 u_r(\text{HFS}) = 0.5 - 0.8 \text{ ppb}$$

$u_r(\alpha)$ [ppb]	Experiment
0.081	Paris (LKB), Rb
0.200	Berkeley, Cs
0.04*	Oxford, Sr
0.5–0.8*	Muonium

*to be performed

Muon anomalous magnetic moment from HFS at MuSEUM

$$a_\mu = \frac{R}{\lambda - R}, \quad R = \frac{\omega_a}{\omega_p}, \quad \lambda = \frac{\mu_\mu}{\mu_p}$$

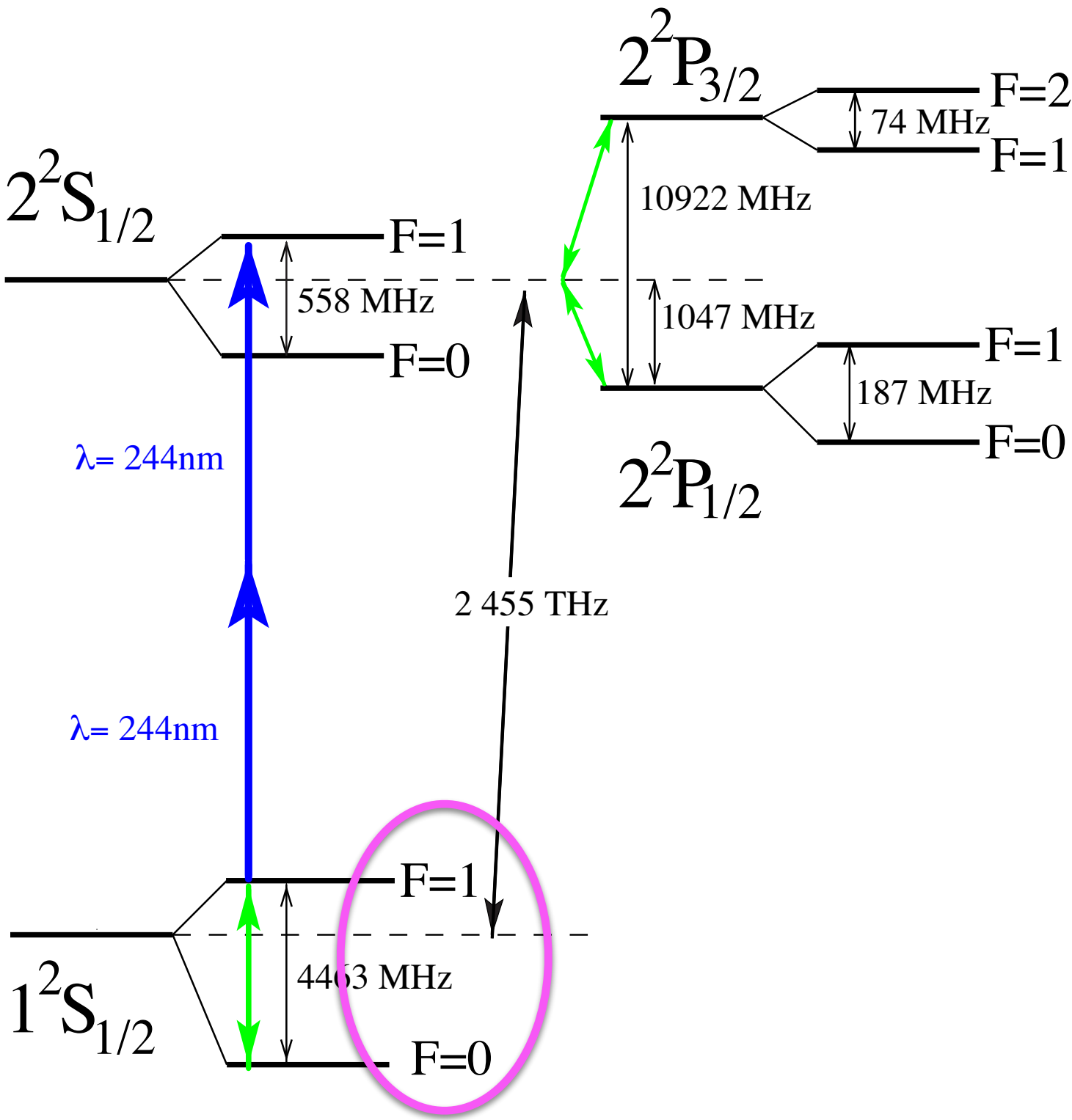
Current experimental precision (LAMPF):

$$u_r(\mu_\mu/\mu_p) = 120 \text{ ppb}$$

$$u_r(\text{HFS}) = 11 \text{ ppb}$$

MuSEUM experiment @ J-PARC:

Contributions	Accuracy	ν_{12} and ν_{34}	$\delta(\Delta\nu_{\text{HFS}})$	$\delta(\mu_\mu/\mu_p)$
Magnetic Field	15 ppb		0.0 ppb	8 ppb
RF power	< 0.1 %	< 1 Hz	0.2 ppb	2 ppb
Kr gas temperature	0.2 deg.	< 2 Hz	0.4 ppb	4 ppb
Kr gas pressure	1 Pa	1 Hz	0.2 ppb	0 ppb
H impurity	< 50 ppm	1 Hz	0.5 ppb	0 ppb
Quadratic dependence		5 Hz	1.0 ppb	5 ppb
Muonium position (x,y)	1 mm	3 Hz	0.6 ppb	6 ppb
Muonium position (z)	1 mm	< 1 Hz	0.2 ppb	2 ppb
Beamline	10 ⁻⁴	< 1 Hz	0.2 ppb	2 ppb
Detector pile-up	w/o absorber w/ absorber	2.8 Hz 0.3 Hz	0.5 ppb < 0.1 ppb	3 ppb < 1 ppb
Total			~ 1.5 ppb	~ 13 ppb



Muon anomalous magnetic moment from HFS at MuSEUM

$$a_\mu = \frac{R}{\lambda - R}, \quad R = \frac{\omega_a}{\omega_p}, \quad \lambda = \frac{\mu_\mu}{\mu_p}$$

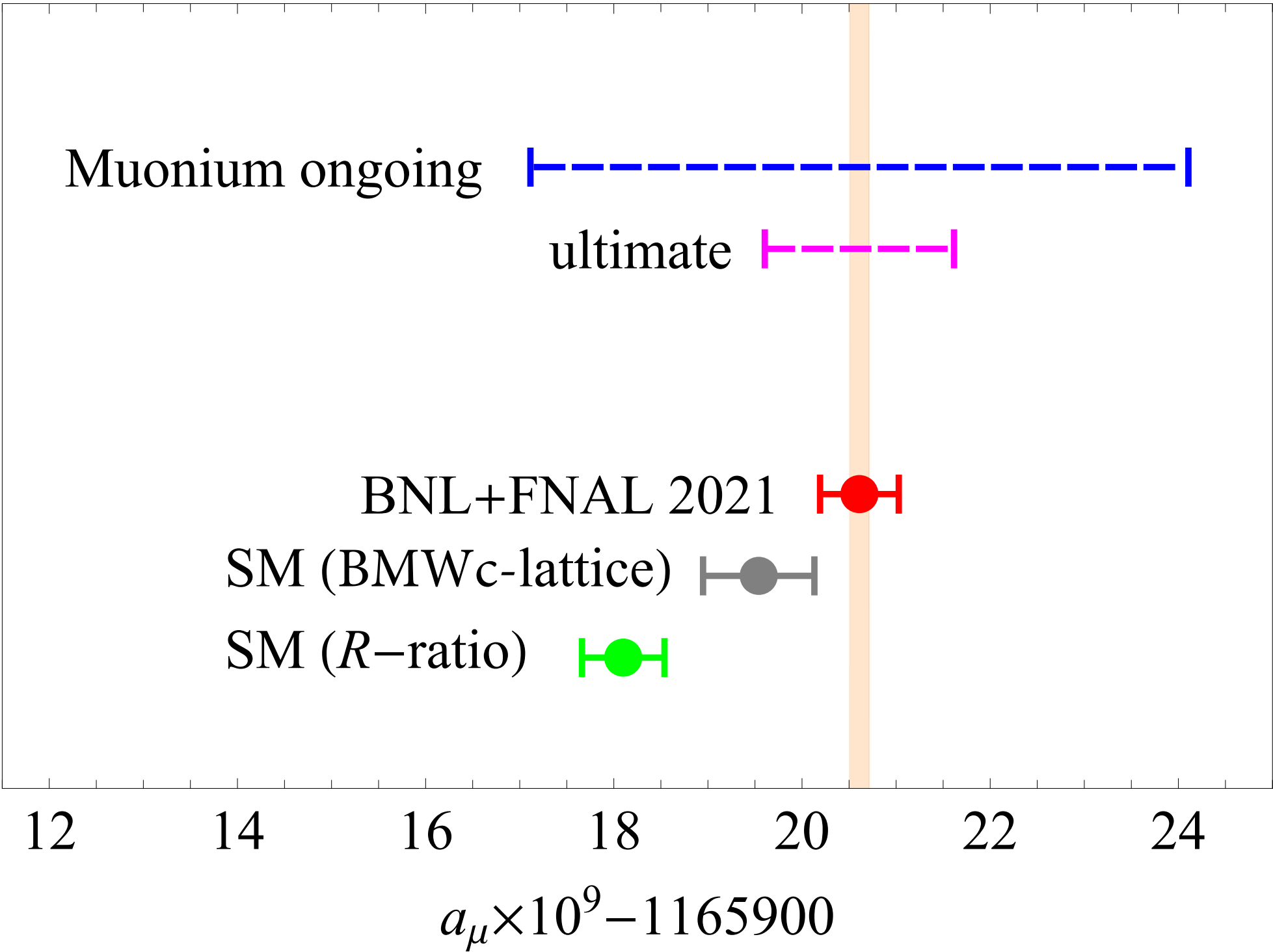
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MuSEUM experiment @ J-PARC:

Contributions	Accuracy	ν_{12} and ν_{34}	$\delta(\Delta\nu_{\text{HFS}})$	$\delta(\mu_\mu/\mu_p)$
Magnetic Field	15 ppb		0.0 ppb	8 ppb
RF power	< 0.1 %	< 1 Hz	0.2 ppb	2 ppb
Kr gas temperature	0.2 deg.	< 2 Hz	0.4 ppb	4 ppb
Kr gas pressure	1 Pa	1 Hz	0.2 ppb	0 ppb
H impurity	< 50 ppm	1 Hz	0.5 ppb	0 ppb
Quadratic dependence		5 Hz	1.0 ppb	5 ppb
Muonium position (x,y)	1 mm	3 Hz	0.6 ppb	6 ppb
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	w/ absorber	0.3 Hz	< 0.1 ppb	< 1 ppb
Total			~ 1.5 ppb	~ 13 ppb



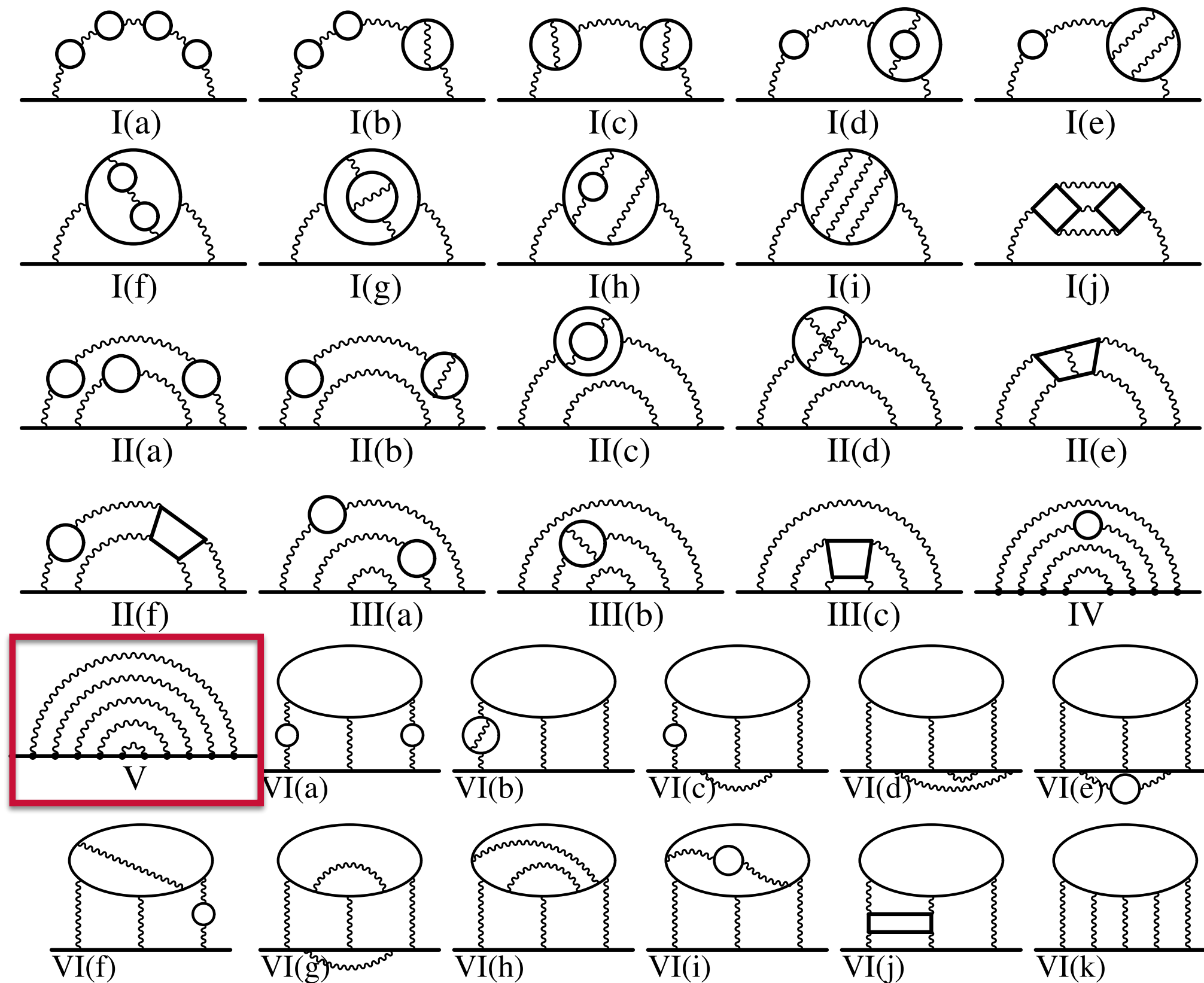
Implications of α_{exp} for Standard Model tests

[Sergey Volkov, Stefano Laporta]

- How relevant is the tension between experimental measurements of α for the Standard Model test based on a_e ?
- What can be done to reduce the error on the QED prediction?
- How urgent is the extension of the QED contribution to 12th (6 loop) order?
- How relevant is the experimental ambiguity in the data-driven evaluation of a_e^{hvp} ?
- What are the implications for the error of the QED and HVP contributions to a_e if the experimental sensitivity of the measurement were improved by an order of magnitude?

10th and 12th order QED contributions

[Sergey Volkov]



$$a_{\ell}^{\text{QED}} = A_1 + A_2\left(\frac{m_{\ell}}{m_{\ell_1}}\right) + A_2\left(\frac{m_{\ell}}{m_{\ell_2}}\right) + A_3\left(\frac{m_{\ell}}{m_{\ell_1}}, \frac{m_{\ell}}{m_{\ell_2}}\right)$$

$$A_i = \sum_{n=1}^{\infty} \left(\frac{\alpha}{\pi}\right)^n A_i^{(2n)}$$

- Discrepancy of 4.6σ observed for **Set V** resolved
- Re-evaluation of 98 self-energy sub-diagrams with higher number of samples in MC integration

[Aoyama, Hayakawa, Kinoshita, Nio, PRL 109 (2012) 111808;

Aoyama, Kinoshita, Nio, Atoms 7 (2019) 28;

Volkov, PRD 100 (2019) 096004; PRD 110 (2024) 036001

Aoyama, Hayakawa, Nio, PRD 111 (2025) L031902]

$$A_1^{(10)} = \begin{cases} 5.891(61) \\ 5.870(128) \end{cases}$$

Volkov 2024

AKHN 2025

$\Rightarrow$

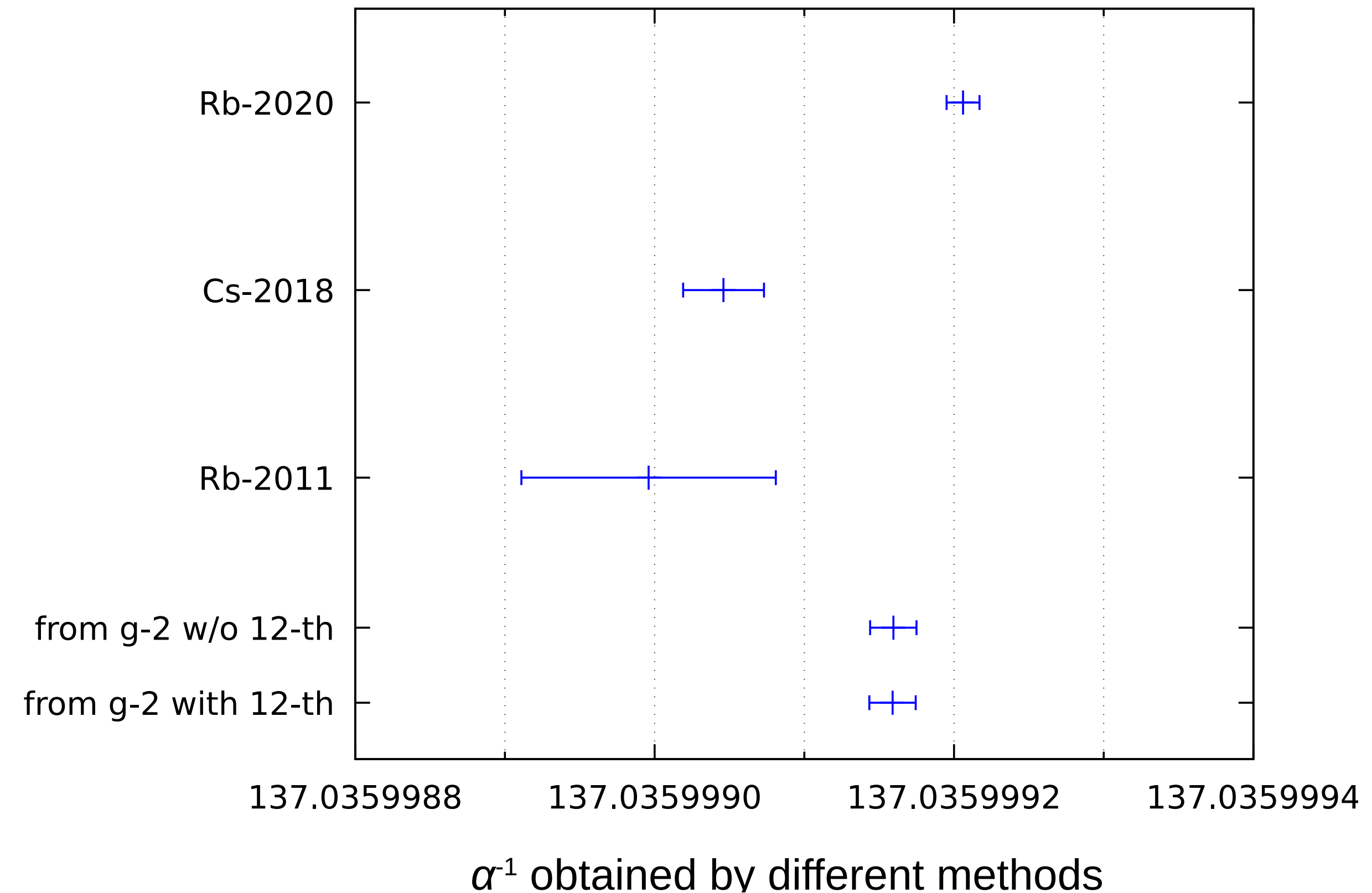
$$A_1^{(10)} = 5.887(55) \quad \text{WP 25}$$

Is the 12th order QED contribution really needed? *[Sergey Volkov]*

Coefficient $A_i^{(2L)}$	#Diagrams	Value
$A_1^{(2)}$	1	0.5
$A_1^{(4)}$	7	$-0.328\,478\,965\,579\,193\dots$
$A_1^{(6)}$	72	$1.181\,241\,456\,587\,200\dots$
$A_1^{(8)}$	891	$-1.912\,245\,764\,926\,445\dots$
$A_1^{(10)}$	12672	5.887(55)

Geometric progression of $A_1^{(2L)} \Rightarrow A_1^{(12)} \simeq -18$

Assume $A_1^{(12)} = -30$



12th order makes no difference*

*given the experimental uncertainty

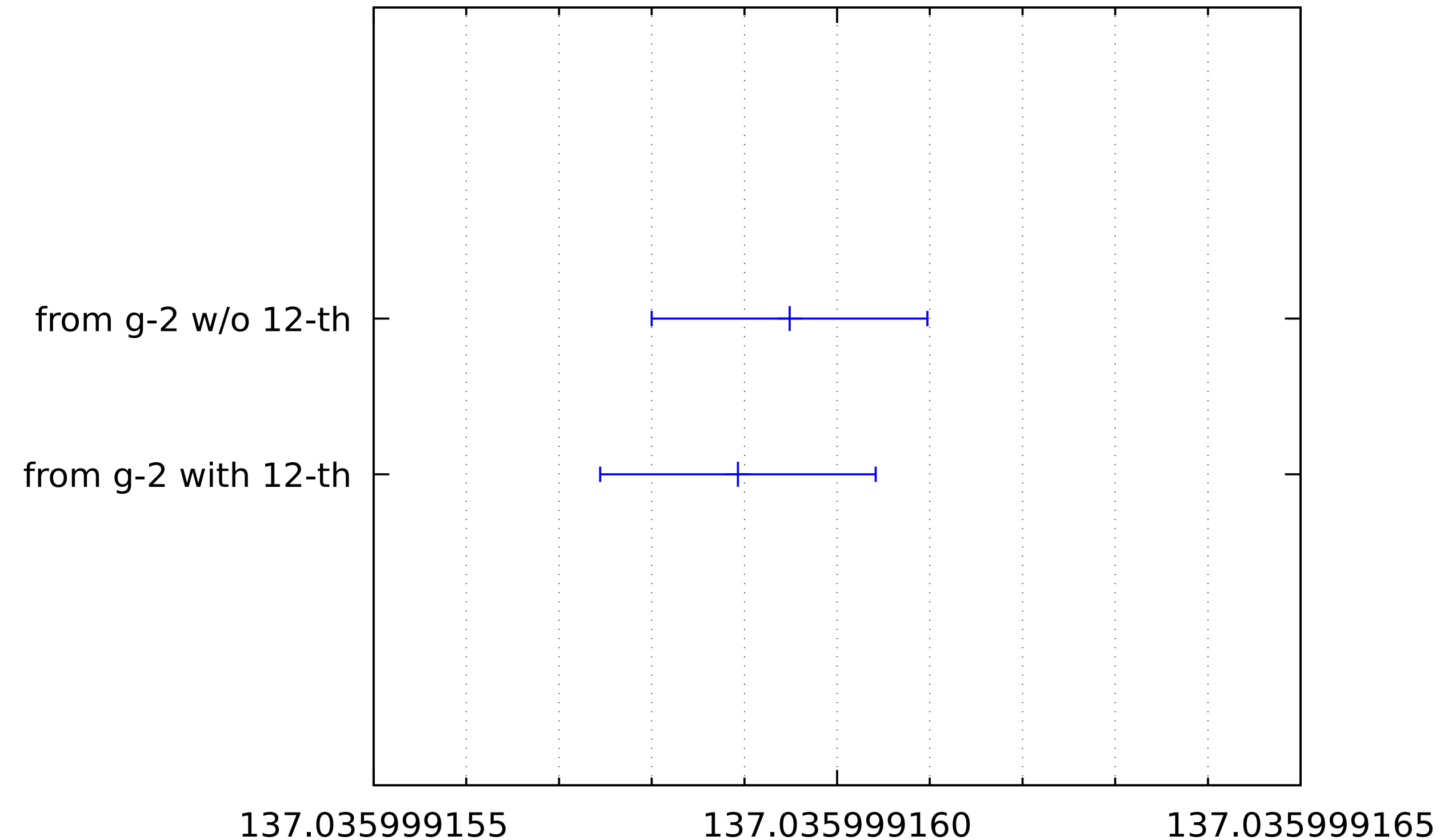
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$A_1^{(10)}$	12672	5.887(55)

Geometric progression of $A_1^{(2L)} \Rightarrow A_1^{(12)} \simeq -18$

Assume $A_1^{(12)} = -30$

+ experimental discrepancy resolved



12th order noticeable but within errors

Is the 12th order QED contribution really needed? *[Sergey Volkov]*

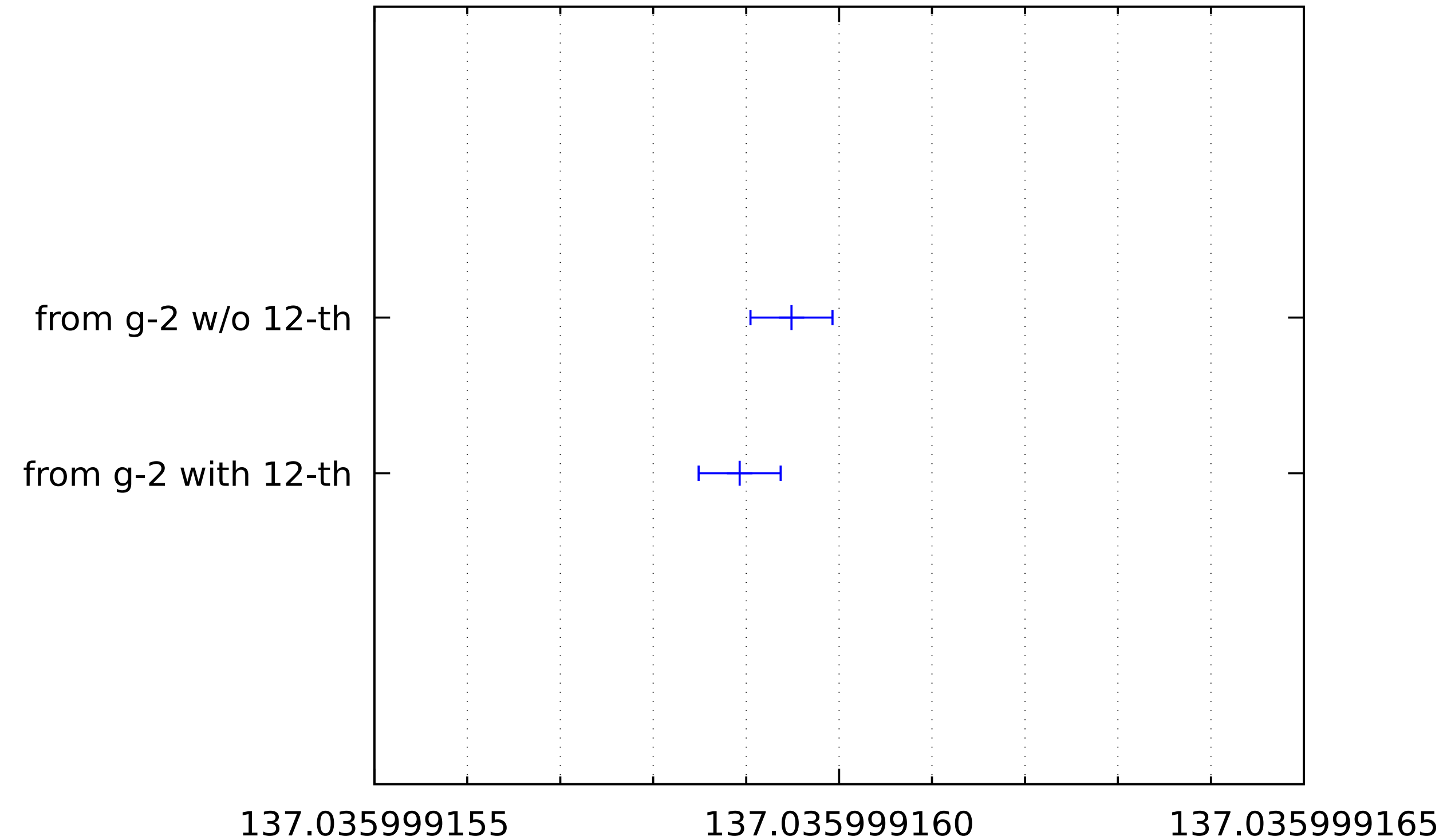
Coefficient $A_i^{(2L)}$	#Diagrams	Value
$A_1^{(2)}$	1	0.5
$A_1^{(4)}$	7	-0.328 478 965 579 193 ...
$A_1^{(6)}$	72	1.181 241 456 587 200 ...
$A_1^{(8)}$	891	-1.912 245 764 926 445 ...
$A_1^{(10)}$	12672	5.887(55)

Geometric progression of $A_1^{(2L)} \Rightarrow A_1^{(12)} \simeq -18$

Assume $A_1^{(12)} = -30$

+ experimental discrepancy resolved

+ no hadronic uncertainty



12th order has visible effect

Is the 12th order QED contribution really needed? *[Sergey Volkov]*

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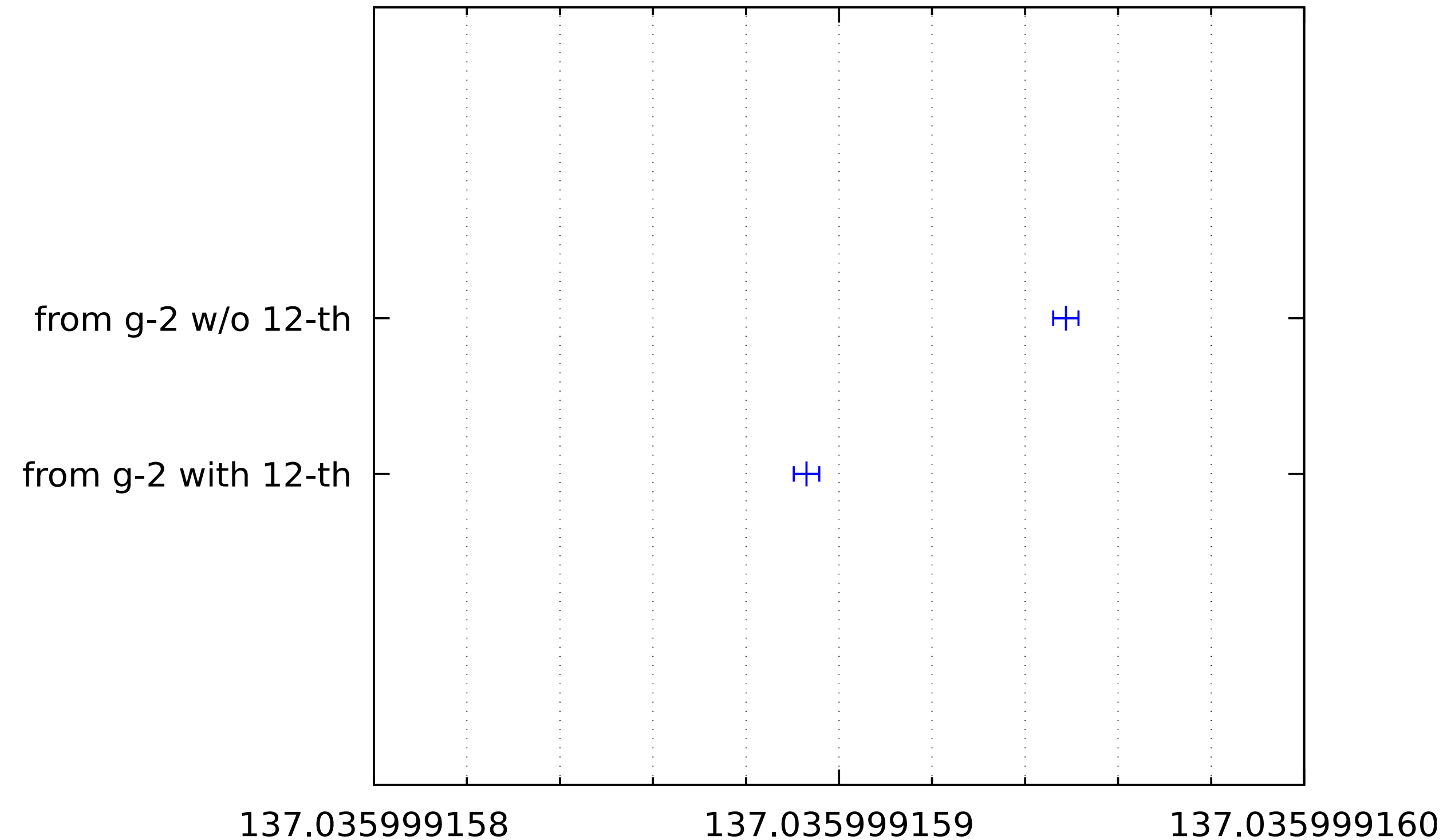
Geometric progression of $A_1^{(2L)} \Rightarrow A_1^{(12)} \simeq -18$

Assume $A_1^{(12)} = -30$

+ experimental discrepancy resolved

+ no hadronic uncertainty

+ no Monte Carlo error in $A_1^{(10)}$



12th order is indispensable!

Calculation of $A_1^{(12)}$ will likely involve Monte Carlo uncertainty

Tackling the 12th order QED contribution

[Sergey Volkov]

Numerical resources required for MC evaluation

order	#diagrams	Integrand code size	GPU time to reach 10% precision
8	891	10 GB	5 min
10	12 672	1 TB	1 month
12	202 770	100 TB*	700 years*
14	3 602 880		
16	70 425 747		

Memory requirements compatible with capacities of current supercomputers, e.g.

JUPITER Booster at FZ Jülich: 24000 NVIDIA GH200 — 3 days running on full machine

Energy consumption: 13 MW — 8 hours of LHC operations

*estimated

Semi-analytical calculation of the 10th order QED contribution *[Stefano Laporta]*

Universal QED contribution at 8th order (4 loops) computed to ~ 1100 digits: *[Laporta 2017]*

$$A_1^{(8)} = -1.9122457649264455741526471674398300540608733 \\ 906587253451713298480060384439806517061427 \dots$$

891 eighth-order Feynman diagrams produce $\sim 10^5$ integrals;

Integrals related by $\sim 10^6$ analytic linear equations (IBP identities) $\rightarrow$ identify Master Integrals

Master Integrals satisfy systems of differential equations (~ 900)

$$\begin{aligned}
A_1^{(8)} = & \frac{1243127611}{130636800} + \frac{30180451}{25920} \zeta(2) - \frac{255842141}{2721600} \zeta(3) - \frac{8873}{3} \zeta(2) \ln 2 + \frac{6768227}{2160} \zeta(4) + \frac{19063}{360} \zeta(2) \ln^2 2 + \frac{12097}{90} \left(a_4 + \frac{1}{24} \ln^4 2 \right) - \frac{2862857}{6480} \zeta(5) - \frac{12720907}{64800} \zeta(3) \zeta(2) \\
& - \frac{221581}{2160} \zeta(4) \ln 2 + \frac{9656}{27} \left(a_5 + \frac{1}{12} \zeta(2) \ln^3 2 - \frac{1}{120} \ln^5 2 \right) + \frac{191490607}{46656} \zeta(6) + \frac{10358551}{43200} \zeta^2(3) - \frac{40136}{27} a_6 + \frac{26404}{27} b_6 - \frac{700706}{675} a_4 \zeta(2) - \frac{26404}{27} a_5 \ln 2 \\
& + \frac{26404}{27} \zeta(5) \ln 2 - \frac{63749}{50} \zeta(3) \zeta(2) \ln 2 - \frac{40723}{135} \zeta(4) \ln^2 2 + \frac{13202}{81} \zeta(3) \ln^3 2 - \frac{253201}{2700} \zeta(2) \ln^4 2 + \frac{7657}{1620} \ln^6 2 + \frac{2895304273}{435456} \zeta(7) + \frac{670276309}{193536} \zeta(4) \zeta(3) + \frac{85933}{63} a_4 \zeta(3) \\
& + \frac{7121162687}{967680} \zeta(5) \zeta(2) - \frac{142793}{18} a_5 \zeta(2) - \frac{195848}{21} a_7 + \frac{195848}{63} b_7 - \frac{116506}{189} d_7 - \frac{4136495}{384} \zeta(6) \ln 2 - \frac{1053568}{189} a_6 \ln 2 + \frac{233012}{189} b_6 \ln 2 + \frac{407771}{432} \zeta^2(3) \ln 2 \\
& - \frac{8937}{2} a_4 \zeta(2) \ln 2 + \frac{833683}{3024} \zeta(5) \ln^2 2 - \frac{3995099}{6048} \zeta(3) \zeta(2) \ln^2 2 - \frac{233012}{189} a_5 \ln^2 2 + \frac{1705273}{1512} \zeta(4) \ln^3 2 + \frac{602303}{4536} \zeta(3) \ln^4 2 - \frac{1650461}{11340} \zeta(2) \ln^5 2 + \frac{52177}{15876} \ln^7 2 \\
& + \sqrt{3} \left[-\frac{14101}{480} \text{Cl}_4 \left(\frac{\pi}{3} \right) - \frac{169703}{1440} \zeta(2) \text{Cl}_2 \left(\frac{\pi}{3} \right) + \frac{494}{27} \text{Im} H_{0,0,0,1,-1,-1} \left(e^{i\frac{\pi}{3}} \right) + \frac{494}{27} \text{Im} H_{0,0,0,1,-1,1} \left(e^{i\frac{2\pi}{3}} \right) + \frac{494}{27} \text{Im} H_{0,0,0,1,1,-1} \left(e^{i\frac{2\pi}{3}} \right) \right. \\
& + 19 \text{Im} H_{0,0,1,0,1,1} \left(e^{i\frac{2\pi}{3}} \right) + \frac{437}{12} \text{Im} H_{0,0,0,1,1,1} \left(e^{i\frac{2\pi}{3}} \right) + \frac{29812}{297} \text{Cl}_6 \left(\frac{\pi}{3} \right) + \frac{4940}{81} a_4 \text{Cl}_2 \left(\frac{\pi}{3} \right) - \frac{520847}{69984} \zeta(5) \pi - \frac{129251}{81} \zeta(4) \text{Cl}_2 \left(\frac{\pi}{3} \right) - \frac{892}{15} \text{Im} H_{0,1,1,-1} \left(e^{i\frac{2\pi}{3}} \right) \zeta(2) \\
& - \frac{1784}{45} \text{Im} H_{0,1,1,-1} \left(e^{i\frac{\pi}{3}} \right) \zeta(2) + \frac{1729}{54} \zeta(3) \text{Im} H_{0,1,-1} \left(e^{i\frac{\pi}{3}} \right) + \frac{1729}{36} \zeta(3) \text{Im} H_{0,1,1} \left(e^{i\frac{2\pi}{3}} \right) + \frac{837190}{729} \text{Cl}_4 \left(\frac{\pi}{3} \right) \zeta(2) + \frac{25937}{4860} \zeta(3) \zeta(2) \pi - \frac{223}{243} \zeta(4) \pi \ln 2 \\
& + \frac{892}{9} \text{Im} H_{0,1,-1} \left(e^{i\frac{\pi}{3}} \right) \zeta(2) \ln 2 + \frac{446}{3} \text{Im} H_{0,1,1} \left(e^{i\frac{2\pi}{3}} \right) \zeta(2) \ln 2 - \frac{7925}{81} \text{Cl}_2 \left(\frac{\pi}{3} \right) \zeta(2) \ln^2 2 + \frac{1235}{486} \text{Cl}_2 \left(\frac{\pi}{3} \right) \ln^4 2 \left. \right] + \frac{13487}{60} \left(\text{Re} H_{0,0,0,1,0,1} \left(e^{i\frac{\pi}{3}} \right) + \text{Cl}_4 \left(\frac{\pi}{3} \right) \text{Cl}_2 \left(\frac{\pi}{3} \right) \right) \\
& + \frac{136781}{360} \text{Cl}_2^2 \left(\frac{\pi}{3} \right) \zeta(2) + \frac{651}{4} \text{Re} H_{0,0,0,1,0,1,-1} \left(e^{i\frac{\pi}{3}} \right) + 651 \text{Re} H_{0,0,0,0,1,1,-1} \left(e^{i\frac{\pi}{3}} \right) - \frac{17577}{32} \text{Re} H_{0,0,1,0,0,1,1} \left(e^{i\frac{2\pi}{3}} \right) - \frac{87885}{64} \text{Re} H_{0,0,0,1,0,1,1} \left(e^{i\frac{2\pi}{3}} \right) \\
& - \frac{17577}{8} \text{Re} H_{0,0,0,0,1,1,1} \left(e^{i\frac{2\pi}{3}} \right) + \frac{651}{4} \text{Cl}_4 \left(\frac{\pi}{3} \right) \text{Im} H_{0,1,-1} \left(e^{i\frac{\pi}{3}} \right) + \frac{1953}{8} \text{Cl}_4 \left(\frac{\pi}{3} \right) \text{Im} H_{0,1,1} \left(e^{i\frac{2\pi}{3}} \right) + \frac{31465}{176} \text{Cl}_6 \left(\frac{\pi}{3} \right) \pi + \frac{211}{4} \text{Re} H_{0,1,0,1,-1} \left(e^{i\frac{\pi}{3}} \right) \zeta(2) \\
& + \frac{211}{2} \text{Re} H_{0,0,1,1,-1} \left(e^{i\frac{\pi}{3}} \right) \zeta(2) + \frac{1899}{16} \text{Re} H_{0,1,0,1,1} \left(e^{i\frac{2\pi}{3}} \right) \zeta(2) + \frac{1899}{8} \text{Re} H_{0,0,1,1,1} \left(e^{i\frac{2\pi}{3}} \right) \zeta(2) + \frac{211}{4} \text{Im} H_{0,1,-1} \left(e^{i\frac{\pi}{3}} \right) \text{Cl}_2 \left(\frac{\pi}{3} \right) \zeta(2) \\
& + \frac{633}{8} \text{Im} H_{0,1,1} \left(e^{i\frac{2\pi}{3}} \right) \text{Cl}_2 \left(\frac{\pi}{3} \right) \zeta(2) - \frac{28276}{25} \zeta(2) \text{Cl}_2 \left(\frac{\pi}{2} \right)^2 + 104 \left(4 \text{Re} H_{0,1,0,1,1} \left(e^{i\frac{\pi}{2}} \right) \zeta(2) + 4 \text{Im} H_{0,1,1} \left(e^{i\frac{\pi}{2}} \right) \text{Cl}_2 \left(\frac{\pi}{2} \right) \zeta(2) - 2 \text{Cl}_4 \left(\frac{\pi}{2} \right) \zeta(2) \pi + \text{Cl}_2^2 \left(\frac{\pi}{2} \right) \zeta(2) \ln 2 \right) \\
& + \sqrt{3} \left[\pi \left(-\frac{28458503}{691200} B_3 + \frac{250077961}{18662400} C_3 \right) + \frac{483913}{77760} \pi f_2(0,0,1) + \pi \left(\frac{4715}{1944} \ln 2 f_2(0,0,1) + \frac{270433}{10935} f_2(0,2,0) - \frac{188147}{4860} f_2(0,1,1) + \frac{188147}{12960} f_2(0,0,2) \right) \right. \\
& + \pi \left(\frac{826595}{248832} \zeta(2) f_2(0,0,1) - \frac{5525}{432} \ln 2 f_2(0,0,2) + \frac{5525}{162} \ln 2 f_2(0,1,1) - \frac{5525}{243} \ln 2 f_2(0,2,0) + \frac{526015}{248832} f_2(0,0,3) - \frac{4675}{768} f_2(0,1,2) + \frac{1805965}{248832} f_2(0,2,1) \right. \\
& - \frac{3710675}{1119744} f_2(0,3,0) - \frac{75145}{124416} f_2(1,0,2) - \frac{213635}{124416} f_2(1,1,1) + \frac{168455}{62208} f_2(1,2,0) + \frac{69245}{124416} f_2(2,1,0) \left. \right] - \frac{4715}{1458} \zeta(2) f_1(0,0,1) + \zeta(2) \left(\frac{2541575}{82944} f_1(0,0,2) \right. \\
& \left. - \frac{556445}{6912} f_1(0,1,1) + \frac{54515}{972} f_1(0,2,0) - \frac{75145}{20736} f_1(1,0,1) \right) - \frac{541}{300} C_{81a} - \frac{629}{60} C_{81b} + \frac{49}{3} C_{81c} - \frac{327}{160} C_{83a} + \frac{49}{36} C_{83b} + \frac{37}{6} C_{83c}.
\end{aligned}$$

(polylogarithms) (harmonic polylogarithms) (elliptic) (unknown elliptic)

Semi-analytical calculation of the 10th order QED contribution

Universal QED contribution at 8th order (4 loops) computed to ~ 1100 digits: *[Laporta 2017]*

$$A_1^{(8)} = -1.9122457649264455741526471674398300540608733 \\ 906587253451713298480060384439806517061427 \dots$$

891 eighth-order Feynman diagrams produce $\sim 10^5$ integrals;

Integrals related by $\sim 10^6$ analytic linear equations (IBP identities) $\rightarrow$ identify Master Integrals

Master Integrals satisfy systems of differential equations (~ 900)

$$A_1^{(8)} = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U \\ U = -\frac{541}{300}C_{81a} - \frac{629}{60}C_{81b} + \frac{49}{3}C_{81c} - \frac{327}{160}C_{83a} + \frac{49}{36}C_{83b} + \frac{37}{6}C_{83c}$$

Apply integer-relation PSLQ algorithm to fit Master Integrals that have no closed-form solution

Floating-point precision required: 1200 digits

Semi-analytical calculation of the 10th order QED contribution

Computational effort

	analytical terms	digits in num.coeff.	num. precision	IBP Systems
4 loop	120	10	1200	10^6
5 loop	5000	20	10^5	10^9

- 30 small 5-loop systems of 5×10^8 IBP identities already solved
- Several TB of memory and tens of TB of fast disk space required
- Runtime scales as $(\text{\#digits})^{3-4}$
 - ⇒ Moderate precision of $O(100)$ digits in $A_1^{(10)}$ realistic, but no analytical fit

Summary

- Resolve the tension between different measurements of α
- Perform a new measurement of a_e with even higher precision
- Perform complementary measurements of α
- Reduce error on 10th order coefficient of QED contribution to a_e
- Tackle the 12th order QED contribution
- Perform state-of-the-art lattice calculation of a_e^{hvp}

Follow-up workshop on a_e and α at Northwestern University in 2–3 years

(offer by Gerald Gabrielse)