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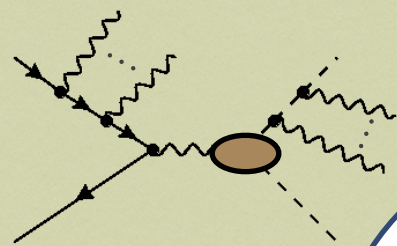
PHOKHARA beyond NLO — an update

William J. Torres Bobadilla
University of Liverpool

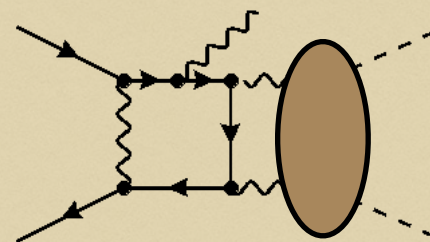
9th Plenary Workshop of the Muon $g-2$ Theory Initiative
Aug 3 – 7, 2026
Department of Physics, University of Connecticut

LEVERHULME
TRUST

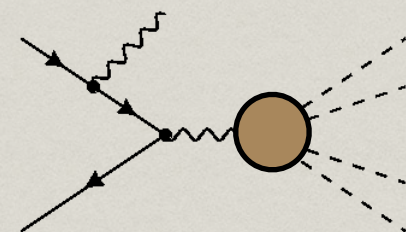
**Soft-photon
Resummation**



Fixed NNLO



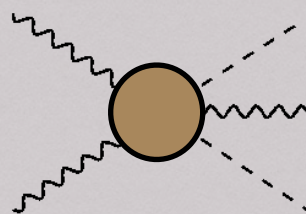
**Radiative Return
Processes**



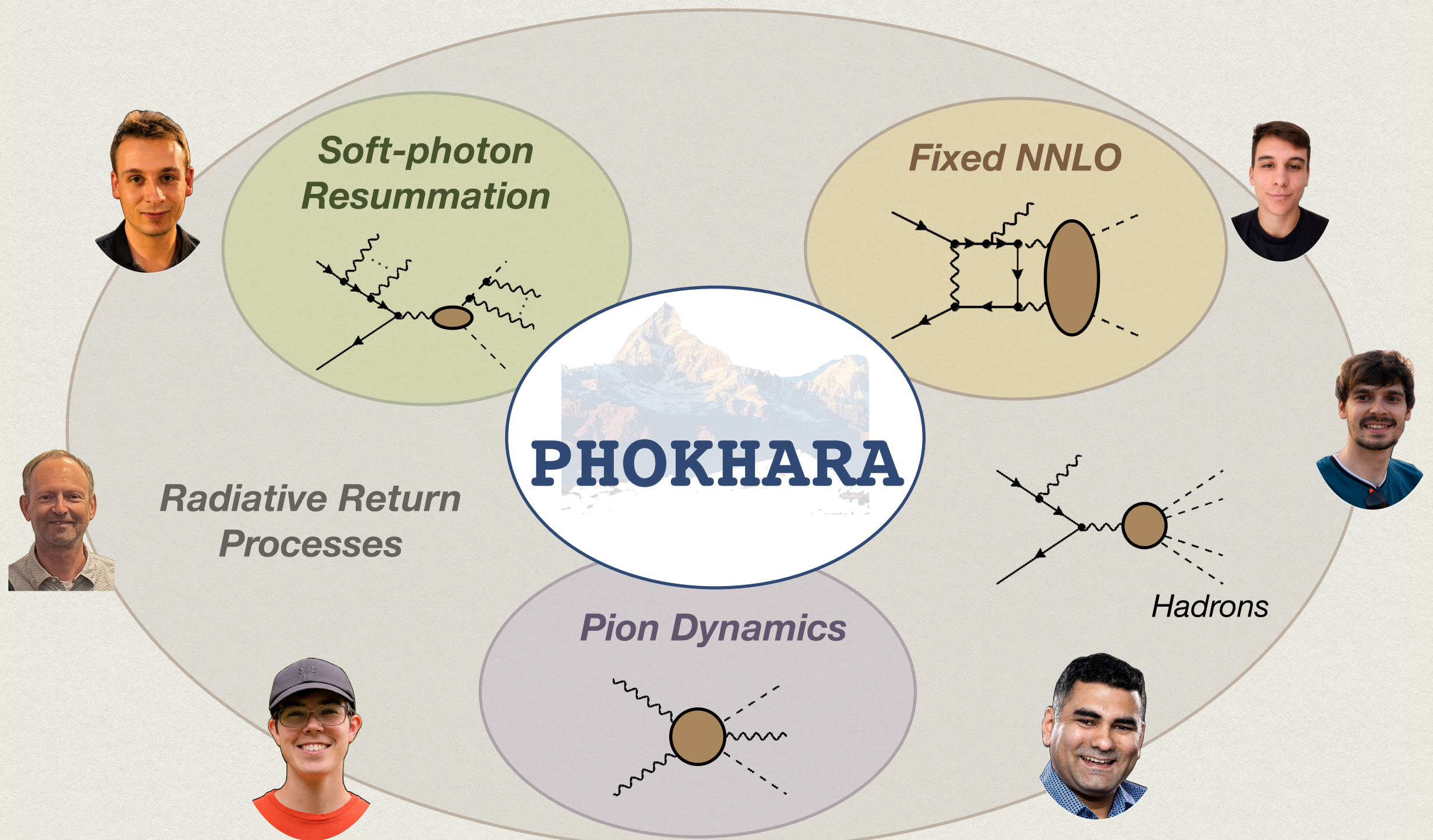
Hadrons

PHOKHARA

Pion Dynamics

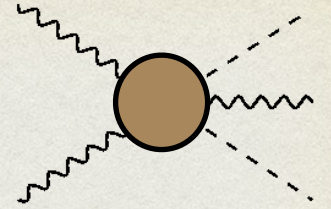


Our theory team



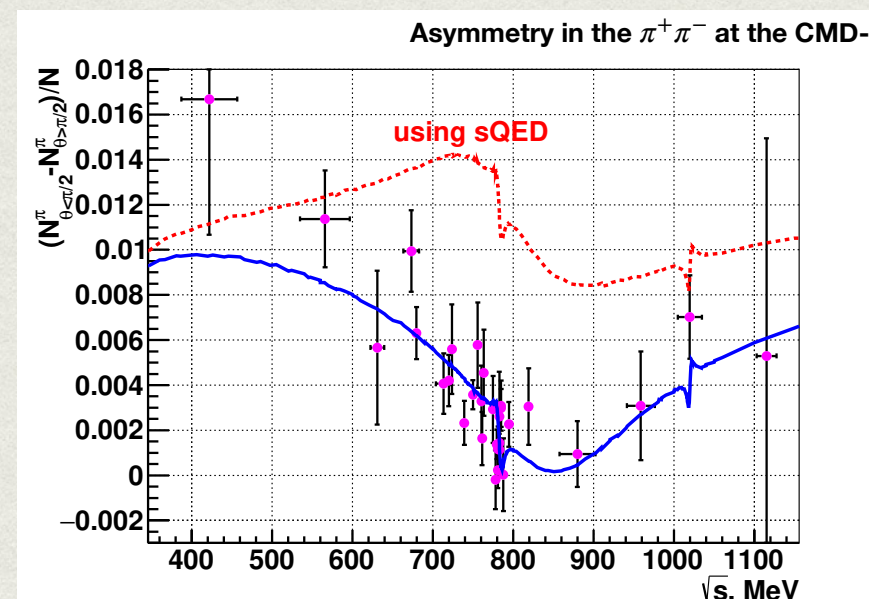
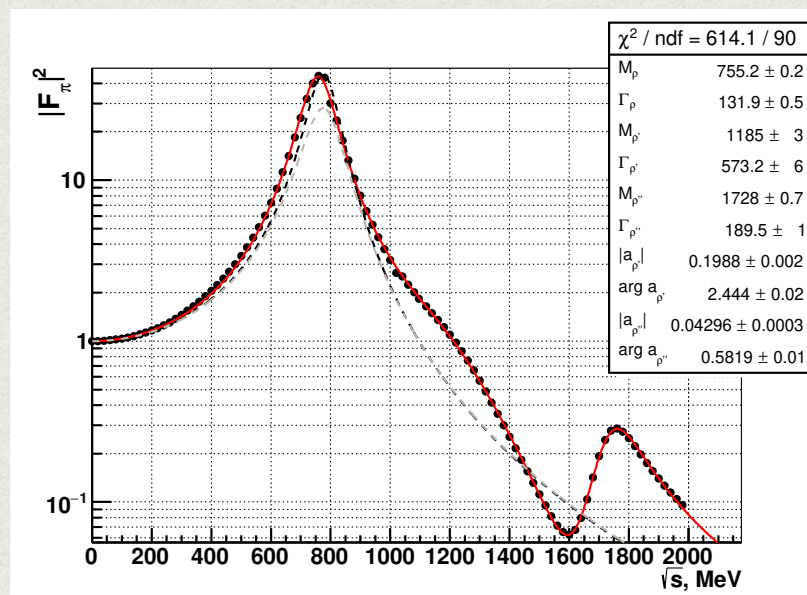
Pion's internal dynamics

Pion's internal dynamics



- Understand structure dependent contributions from the pion's internal dynamics.
- Need of a form factor that accounts for non-perturbative interactions.

[Ignatov, Lee (2022)]



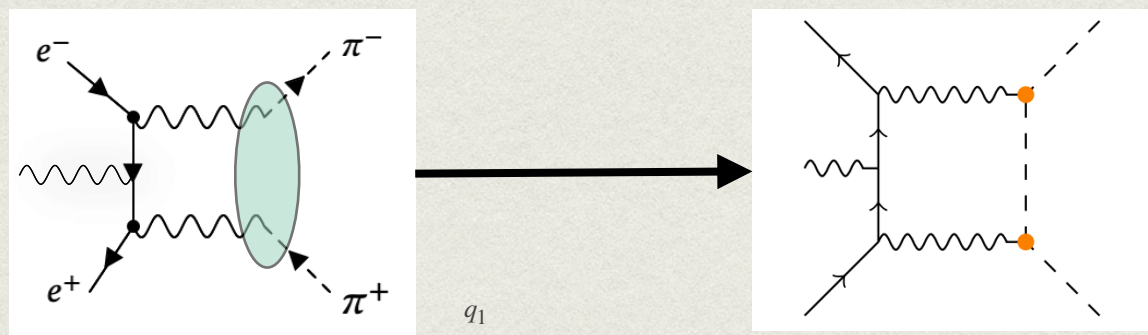
Pion form factor expressed in terms of sum of the Gounaris-Sakurai resonance parametrisation

- Go beyond $F \times \text{sQED} \rightarrow (\text{GVMD}, \text{FsQED}, \dots)$.
- Our goal: radiative return processes w/n PHOKHARA.

(Fit the FF to a sum of Breit-Wigner functions)

Generalised vector-meson dominance

- PHOKHARA10 employs $F \times s\text{QED}$ to describe Pion's internal dynamics
- Insertion of GVMD in scattering processes



Amplitudes are *analytically* constructed by following a purely diagrammatic approach:
qgraph $\rightarrow$ **Form** $\rightarrow$ **Collier**

$$q^\mu \text{ wavy line} \rightarrow \text{wavy line} = i Q_\pi (p_1 - p_2)^\mu F_\pi(q^2)$$

$$q_1^\mu \text{ wavy line} \rightarrow \text{wavy line} = 2i Q_\pi^2 g^{\mu\nu} F_\pi(q_1^2) F_\pi(q_2^2)$$

everything relies on modified sQED Feynman rules

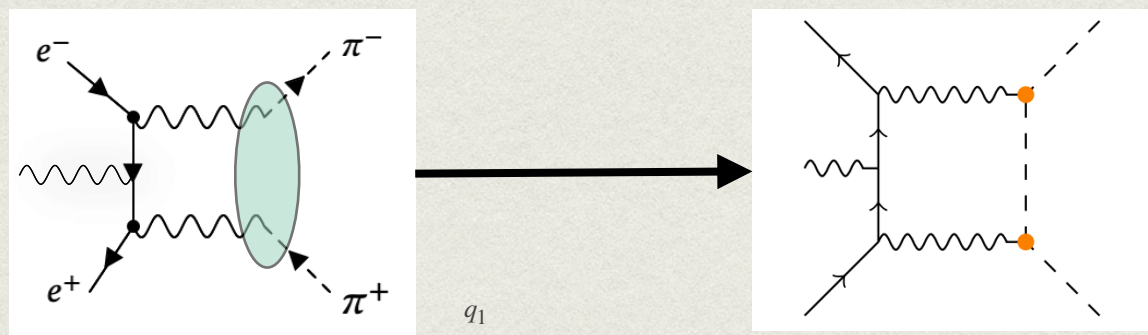
$$F_\pi(q^2) = 1 + \sum_v a_v \frac{q^2}{\Lambda_v - q^2}$$

Suitable for any form factor parametrisation

(our choice: fit the FF to a sum of Breit-Wigner functions)

Generalised vector-meson dominance

- PHOKHARA10 employs $F \times s\text{QED}$ to describe Pion's internal dynamics
- Insertion of GVMD in scattering processes



Amplitudes are *analytically* constructed by following a purely diagrammatic approach:
qgraph → **Form** → **Collier**

$$\begin{aligned}
 q^\mu \text{ wavy} \rightarrow \text{wavy} &= i Q_\pi (p_1 - p_2)^\mu F_\pi(q^2), \\
 q_1^\mu \text{ wavy} \rightarrow \text{wavy} &= 2i Q_\pi^2 g^{\mu\nu} F_\pi(q_1^2) F_\pi(q_2^2)
 \end{aligned}$$

everything relies on modified sQED Feynman rules

$$F_\pi(q^2) = 1 + \sum_v a_v \frac{q^2}{\Lambda_v - q^2}$$

GVMD $\gamma^* \gamma^* \rightarrow \pi^+ \pi^- n\gamma$ with n on-shell photons

$$\begin{aligned}
 &\gamma^* \text{ wavy} \rightarrow \pi^- \text{ dashed} \\
 &\gamma^* \text{ wavy} \rightarrow \pi^+ \text{ dashed} \\
 &\text{GVMD} \\
 &= \underbrace{\text{sQED}}_{\delta_{NN}} - \sum_v a_v \left(\underbrace{\text{sQED}}_{\Lambda_v \text{ wavy}} + \underbrace{\text{sQED}}_{\Lambda_v \text{ wavy}} \right) + \sum_{v,w} a_v a_w \underbrace{\text{sQED}}_{\Lambda_v \text{ wavy}, \Lambda_w \text{ wavy}} \\
 &\hspace{15em} \delta_{MN}(\Lambda_v) \hspace{15em} \delta_{MM}(\Lambda_v, \Lambda_w)
 \end{aligned}$$

GVMD in KLOE large and small angle scenarios

[Petit Rosas, Shekhovtsova, WJT (2026)]

Large angle

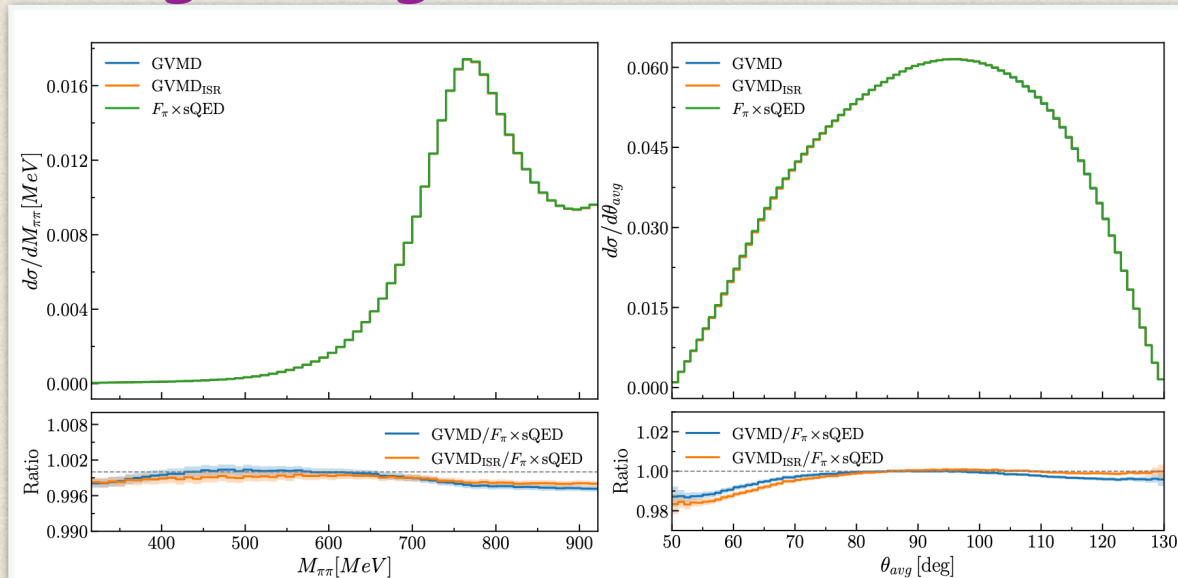


Figure 2: Differential cross section versus pion-pair invariant mass (left) and θ_{avg} (right) under $F_\pi \times \text{sQED}$, the full GVMD modifications and only the ISR diagrams modified with GVMD. The bottom panel of each plot shows, the ratio between the modifications and $F_\pi \times \text{sQED}$. Colour bands correspond to statistical uncertainties. Note the difference between y-axis scales.

$$\begin{aligned} \sqrt{s} &= 1.02 \text{ GeV}, \quad 50^\circ \leq \theta_\pm \leq 130^\circ, \\ E_\gamma &> 20 \text{ MeV}, \quad 50^\circ \leq \theta_\gamma \leq 130^\circ, \\ |p_z^\pm| &> 90 \text{ MeV} \quad \vee \quad p_\perp^\pm > 160 \text{ MeV}, \\ 0.1 \text{ GeV}^2 &\leq M_{\pi\pi}^2 \leq 0.85 \text{ GeV}^2. \end{aligned}$$

Small angle

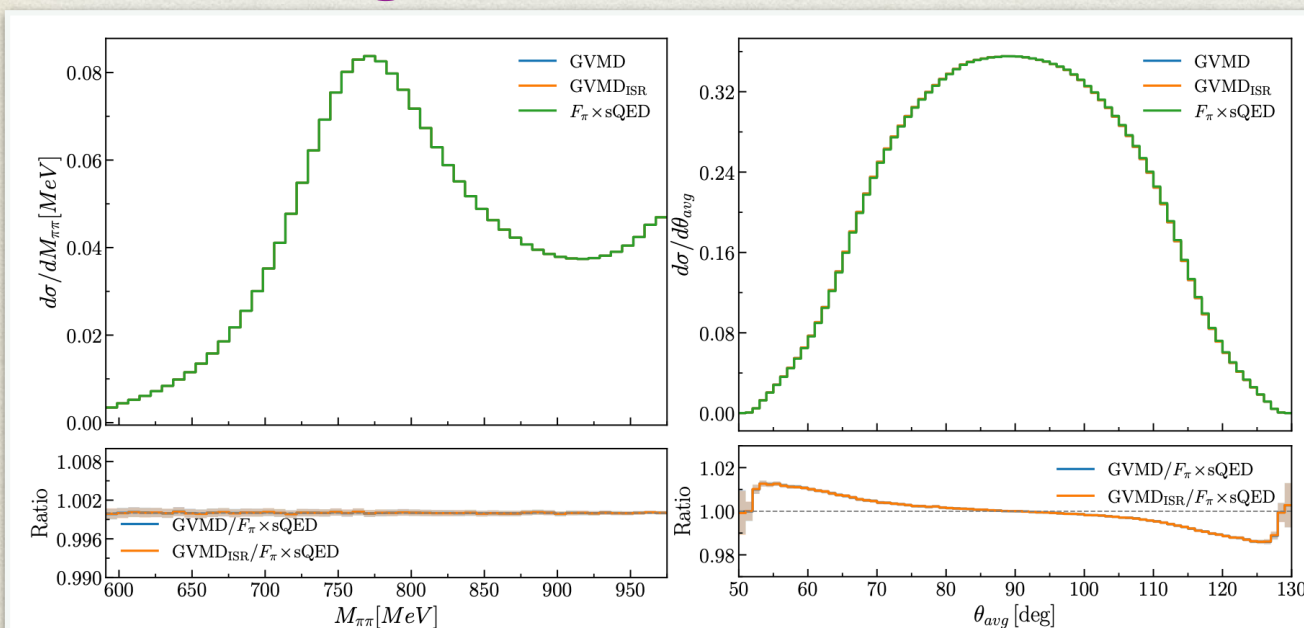


Figure 3: Differential cross section versus pion-pair invariant mass (left) and θ_{avg} (right) under $F_\pi \times \text{sQED}$, the full GVMD modifications and only the ISR diagrams modified with GVMD, under the KLOE-like small angle scenario. The ratio between the GVMD modifications and $F_\pi \times \text{sQED}$ can be seen in the bottom of the plot.

$$\begin{aligned} \sqrt{s} &= 1.02 \text{ GeV}, \quad 50^\circ \leq \theta_\pm \leq 130^\circ, \\ |p_z^\pm| &> 90 \text{ MeV} \quad \vee \quad p_\perp^\pm > 160 \text{ MeV}, \\ \theta_\gamma &\leq 15^\circ \quad \vee \quad \theta_\gamma > 165^\circ, \\ 0.35 \text{ GeV}^2 &\leq M_{\pi\pi}^2 \leq 0.95 \text{ GeV}^2, \end{aligned}$$

GVMD in Forward–backward asymmetry in KLOE

[Petit Rosas, Shekhovtsova, WJT (2026)]

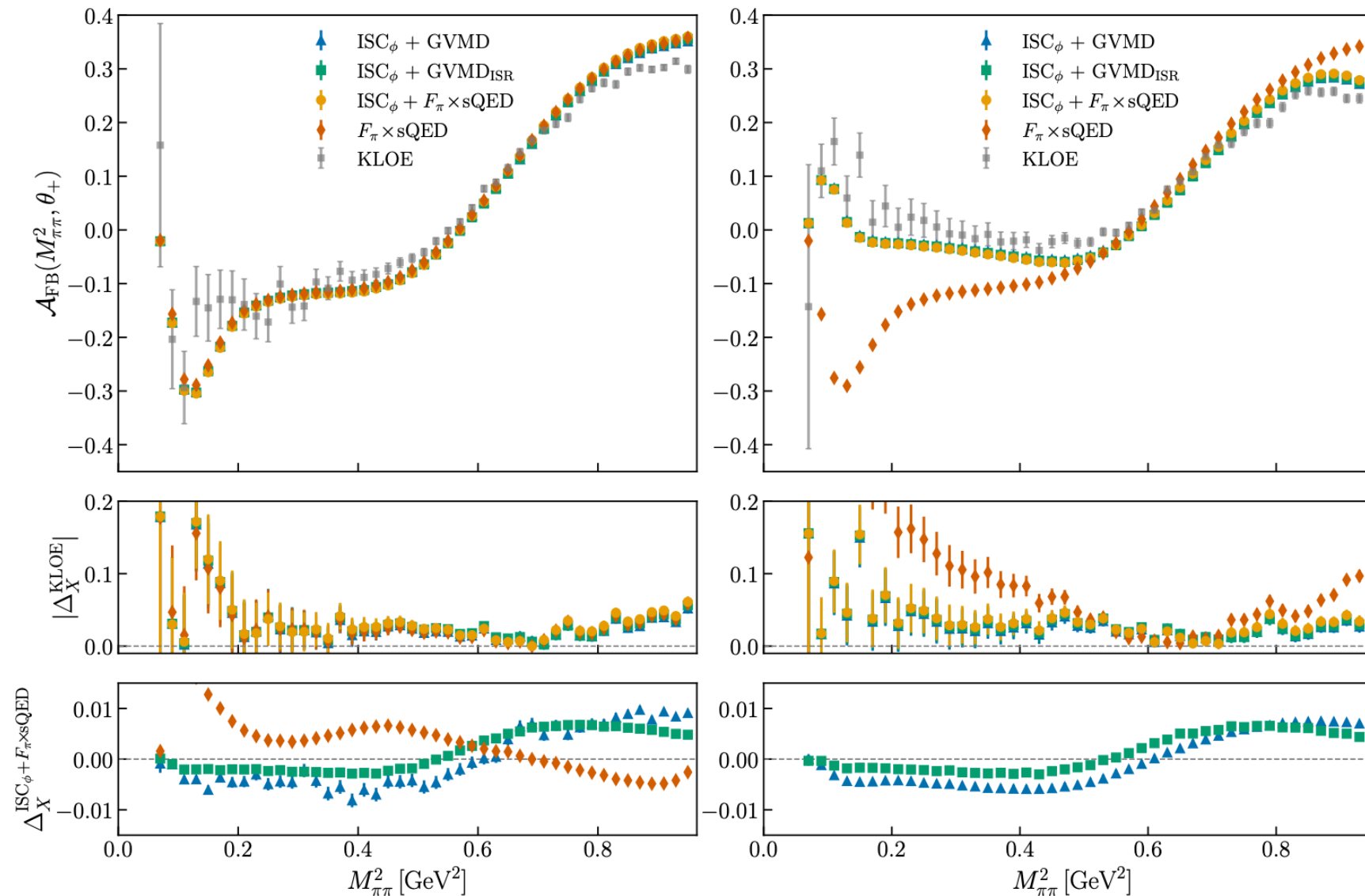


Figure 5: In the top panels we show the forward–backward asymmetry for a relaxed LA KLOE selection at $\sqrt{s} = 1$ GeV (left) and $\sqrt{s} = 1.02$ GeV (right). The KLOE experimental data is shown in grey, while the coloured points correspond to *Phokhara* predictions including different contributions (see text for details). The middle panels display the absolute difference between data and each prediction, $\Delta_X^{KLOE} = KLOE - X$. The bottom panels compare the MC approaches: for the left plot we take $F_\pi \times sQED$ as the reference, while for the right plot we use $F_\pi \times sQED + ISC_\phi$.

$50^\circ \leq \theta_{\gamma, \pm} \leq 130^\circ,$
 $|M_{\text{trk}} - M_\pm| < 10 \text{ MeV}.$

GVMD + structure dependent $\gamma^* \rightarrow \pi^+ \pi^- \gamma$

[Shekhovtsova, Venanzoni, Pancheri (2009)]

Radiative return processes beyond NLO

Towards NNLO

Anatomy @ NNLO

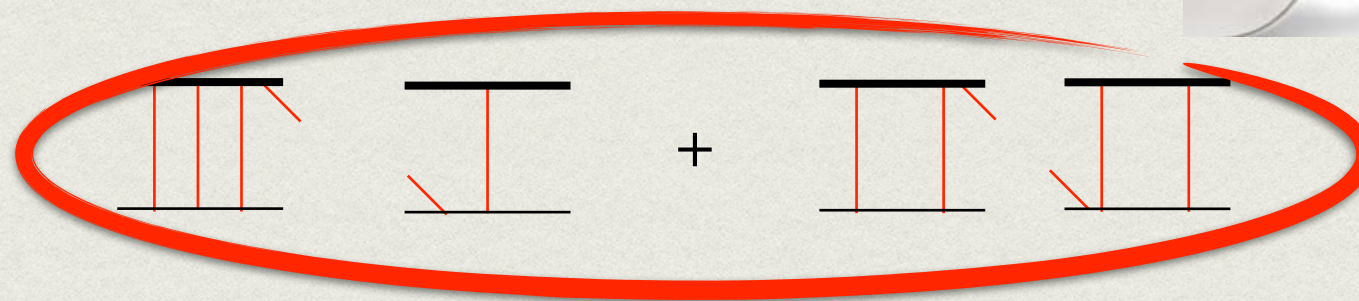
- Real-Real contribution
Tree-level $(n+2)$ -particles



- Real-Virtual Contribution
one-loop $(n+1)$ -particles



- Virtual-Virtual Contribution
two-loop n -particles



- ☑ For radiative return processes:

$2 \rightarrow 3$ scattering processes :: Kinematic invariants $(s_{12}, s_{23}, s_{34}, s_{45}, s_{51}, m_e^2, m_F^2)$.

- ☑ One-loop contributions to NLO theoretical predictions:
straightforward with automated tools!

- ☐ Full NNLO theoretical predictions:

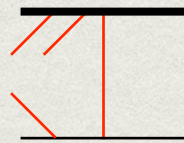
- ☑ Extend numerical evaluation of one-loop scattering amplitudes [Dave, Paltrinieri, Petit Rosas, WJT (2026)]

- ☑ Evaluation of two-loop Feynman [Pozzoli, WJT (2026)] + [Dave, Matijasic, Melnichenko, Petit, Pozzoli, WJT, Weinzierl (in progress)]

Towards NNLO

Anatomy @ NNLO

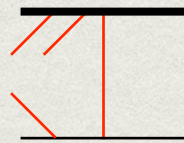
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Towards NNLO

Anatomy @ NNLO

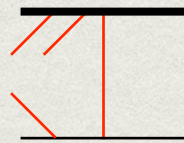
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Towards NNLO

Anatomy @ NNLO

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Towards NNLO

Anatomy @ NNLO

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two-loop n -particles



Soft-photon resummation

- ✓ Approximating NNLO contributions can probe dominant effects coming from radiative corrections.
- ✓ Account for soft phase-space regions
- ✓ The route:
Yennie, Frauschi, Suura exponentiation
+ *Coherent Exclusive Exponentiation*



Soft-photon resummation

○ YFS exponentiation

[Yennie, Frauschi, Suura (1961)]

$$\sigma_{\text{NLO}}^{\text{resum}} = \sum_{n_\gamma} \int_{\Omega} d\text{LIPS}_{n_\gamma+2} \frac{e^{\alpha Y_{\text{FF}}}}{n_\gamma!} \sum_{\text{hel}} \left| \sum_{\phi \in [I,F]^{n_\gamma}} \left(\prod_{i=1}^{n_\gamma} S_i(k_i) \Theta(k_i, \Omega) \right) \left(\beta_0 + \sum_{i=1}^{n_\gamma} \frac{\beta_1(k_i)}{S(k_i)} + \sum_{\substack{i,j=1 \\ j < k}}^{n_\gamma} \frac{\beta_1(k_i, k_j)}{S(k_i) S(k_j)} \right) \right|^2$$

Partitions (ISR/FSR) Regularised amplitudes
Eikonal factors Resolved domain

- * Virtual & real amplitudes inside β 's
- * No approximations — Control on ISR and FSR contributions!
- * Need momenta mappings to map N -photon events to i -photon events for β_i .

○ YFS form factor

$$Y_{\text{FF}}(\Omega) = 2\alpha \sum_{i < j} \Re \left(\textcolor{red}{B}(p_i, p_j) + \textcolor{blue}{\tilde{B}}(p_i, p_j, \Omega) \right) \quad \text{fermions and hadrons dipoles}$$

Virtual Real photon emitted in the unresolved domain Ω

All calculations are performed in **dimensional regularisation** and in a **Lorentz-invariant way**

○ Regularised amplitudes

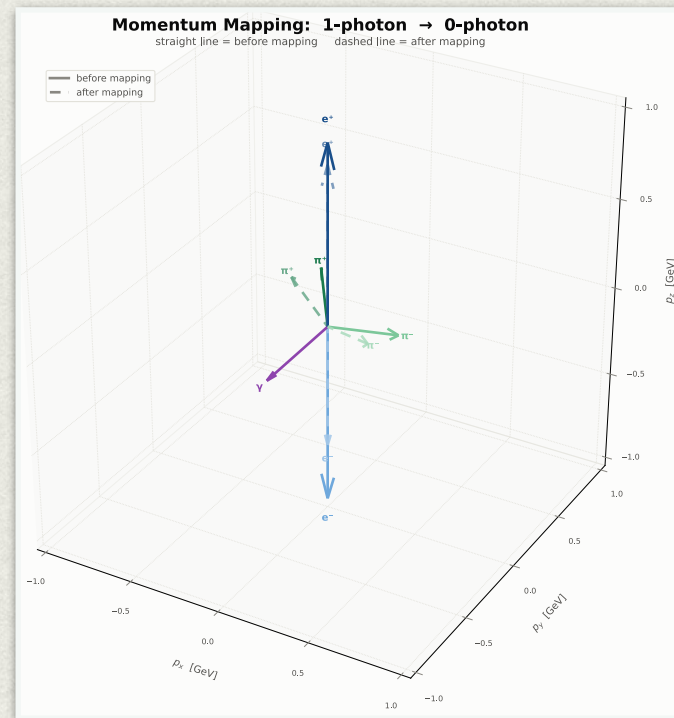
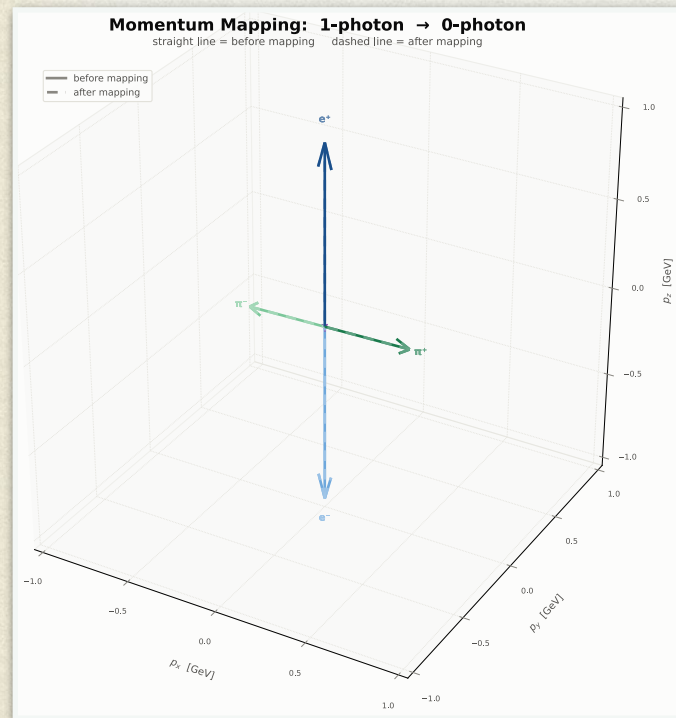
- * Use form factor decomposition for the calculation of amplitudes
 $e^+e^- \rightarrow \pi^+\pi^-\gamma(\gamma)$ and $e^+e^- \rightarrow \mu^+\mu^-\gamma(\gamma)$

[Dave, Paltrinieri, Petit Rosas, WJT (2026)]

Momenta mappings

[Provided by J. Paltrinieri & L. Flower]

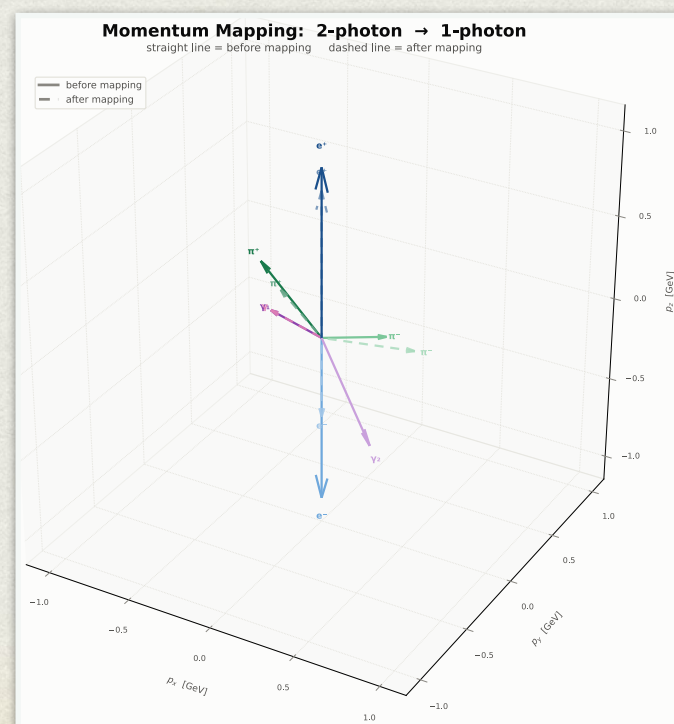
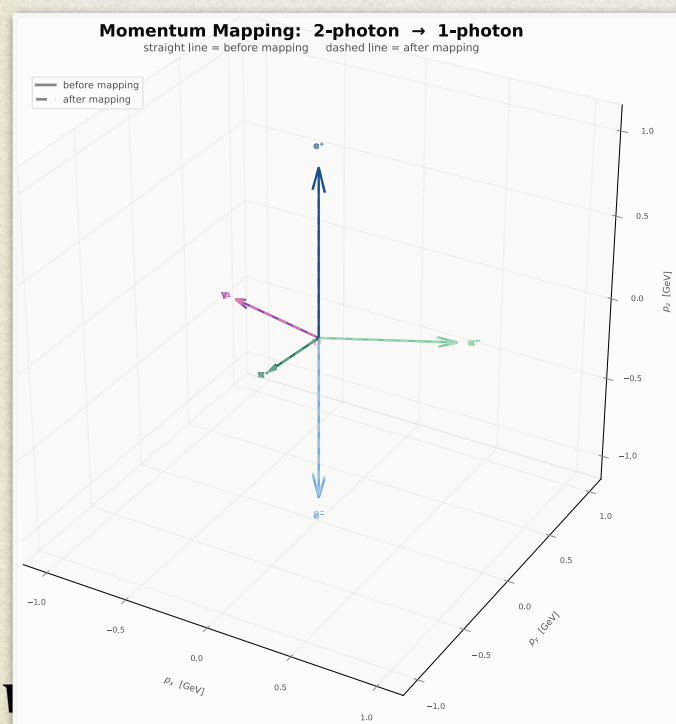
Map N -photon events to i -photon events for reg. ampls.



Scan mode:

$$1L: 2 \rightarrow 2 + 0\gamma$$

$$0L: 2 \rightarrow 2 + 1\gamma$$



Radiative return mode:

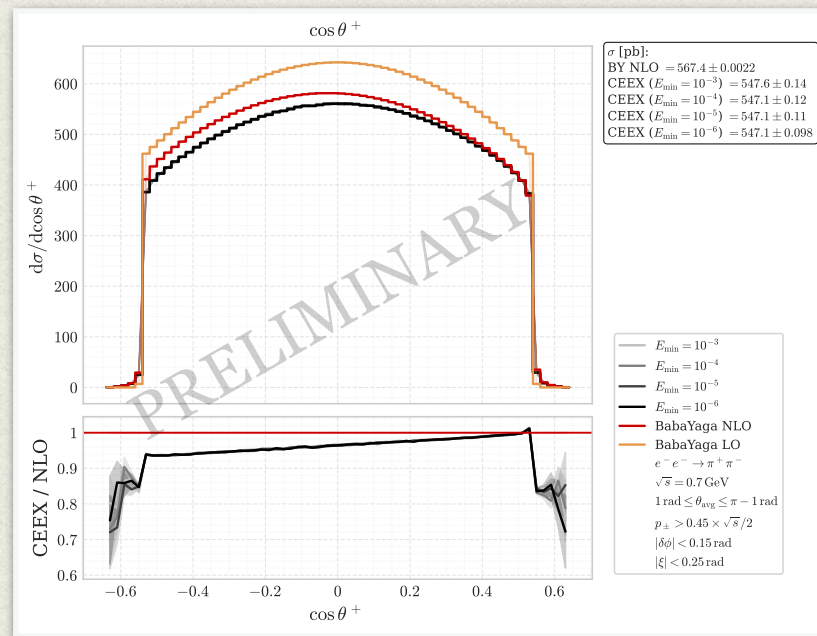
$$1L: 2 \rightarrow 2 + 1\gamma$$

$$0L: 2 \rightarrow 2 + 2\gamma$$

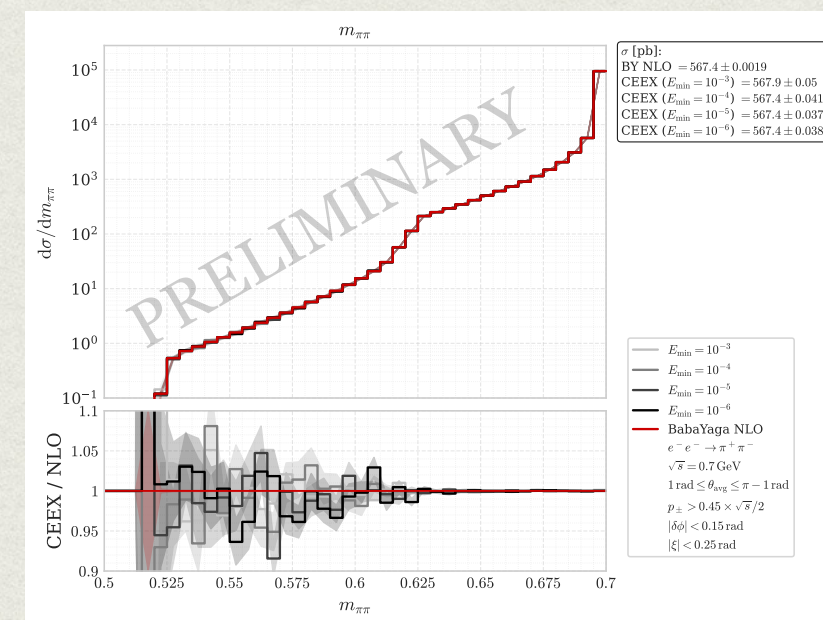
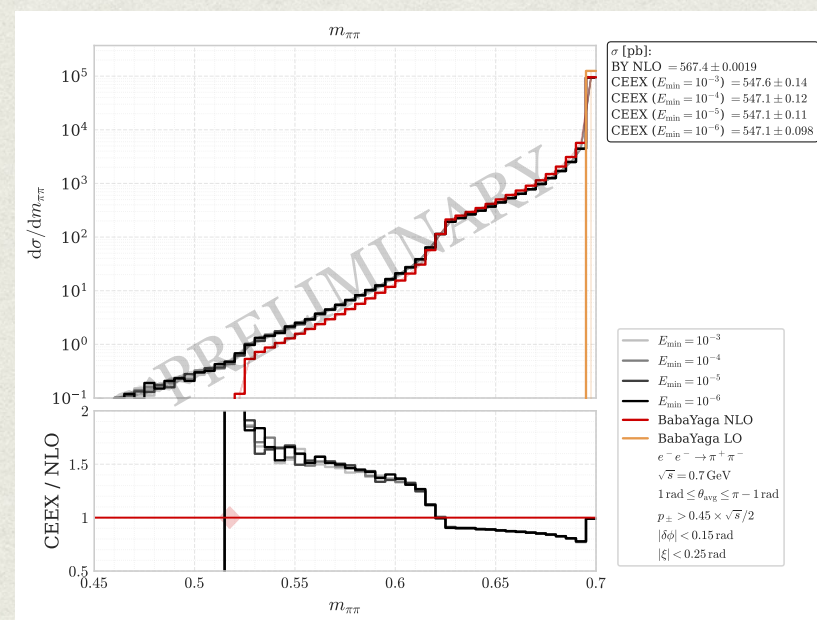
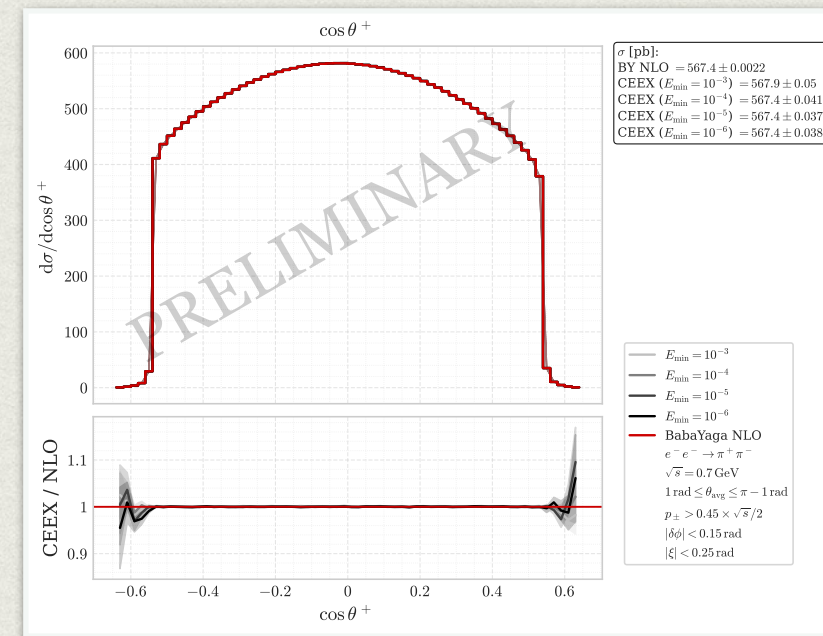
Is YFS a way to go?

- Scan mode (for the CMD-3 scenario)

LO resummed (3 radiated photons)



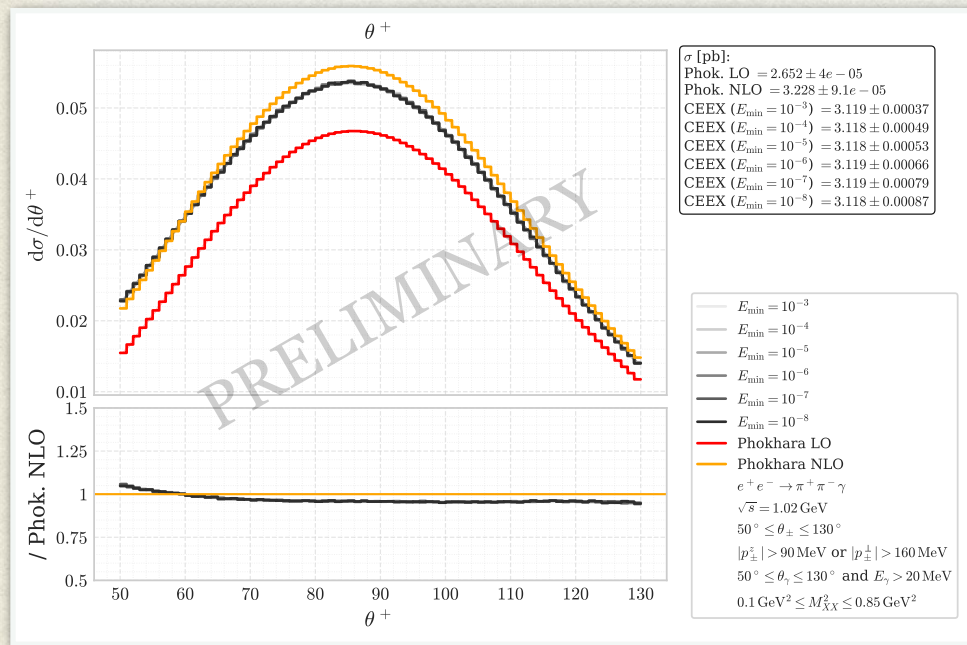
Fixed NLO (truncated at a desirable α)



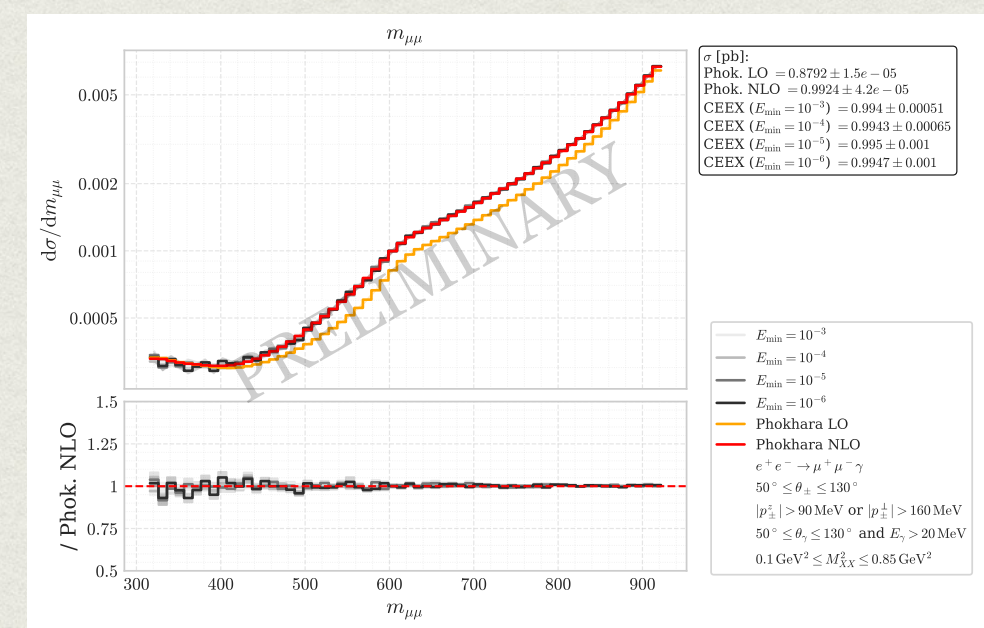
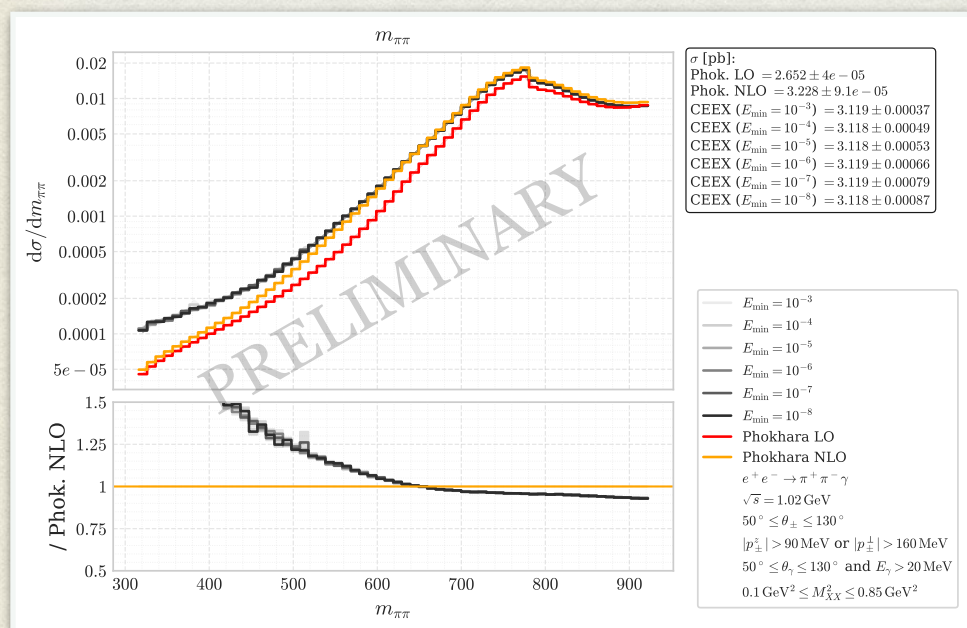
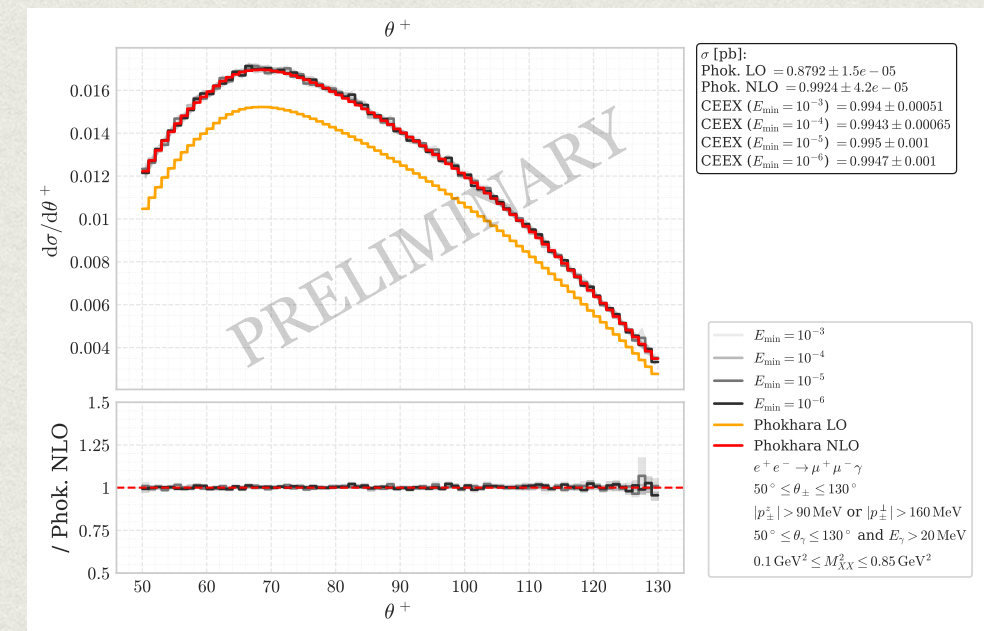
Toward NLO+ for radiative return processes

- Large-angle ISR-tagged for $e^+e^- \rightarrow \mu^+\mu^-\gamma(\gamma)$

LO resummed (2 radiated photons)



Fixed NLO (truncated at a desirable α)



Radiative return processes at fixed NNLO

$$e^+e^- \rightarrow F^+F^-\gamma \text{ @ NNLO}$$

Anatomy @ NNLO

- Real-Real contribution
Tree-level $(n+2)$ -particles



- Real-Virtual Contribution
one-loop $(n+1)$ -particles



- Virtual-Virtual Contribution
two-loop n -particles

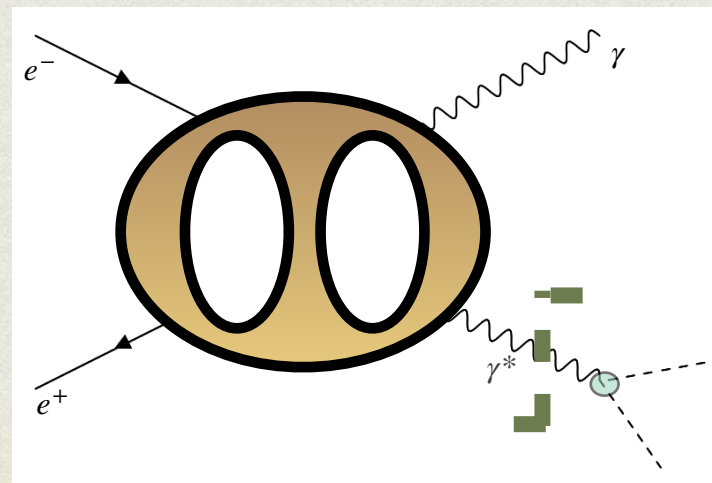


- Initial state contributions

$$e^+e^- \rightarrow \gamma \gamma^* \rightarrow \mu^+\mu^-$$

or

$$e^+e^- \rightarrow \gamma \gamma^* \rightarrow \pi^+\pi^-$$



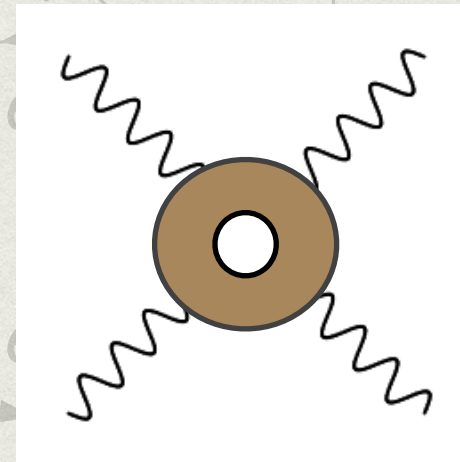
$$e^+e^- \rightarrow \gamma\gamma^* @ 2L$$

- ✓ Decomposition of the two-loop amplitudes (from leptonic to hadronic corrections)

$$\mathcal{A}^{(2)} = A^{(2)} + N_F B^{(2)}$$



Photon correction



Light-by-light

- ✓ Amplitudes in terms of **12** independent tensor structures in $D = 4$ dimensions

$$\mathcal{T}_1 = v(p_2)\bar{u}(p_1)(\varepsilon_3^* \cdot \varepsilon_4^*),$$

$$\mathcal{T}_2 = v(p_2)\bar{u}(p_1)(p_1 \cdot \varepsilon_3^*)(p_1 \cdot \varepsilon_4^*),$$

$$\mathcal{T}_3 = v(p_2)\bar{u}(p_1)(p_1 \cdot \varepsilon_3^*)(p_2 \cdot \varepsilon_4^*),$$

$$\mathcal{T}_4 = v(p_2)\varepsilon_4^* \bar{u}(p_1)(p_1 \cdot \varepsilon_3^*),$$

⋮

$$2^2 \times 2 \times 3 = 24/2 = 12$$

2 polarisation for each leptons

2 helicity states for on-shell photon

3 polarisation states for off-shell photon

parity conservation

- Two-loop needs evaluation of Feynman integrals!

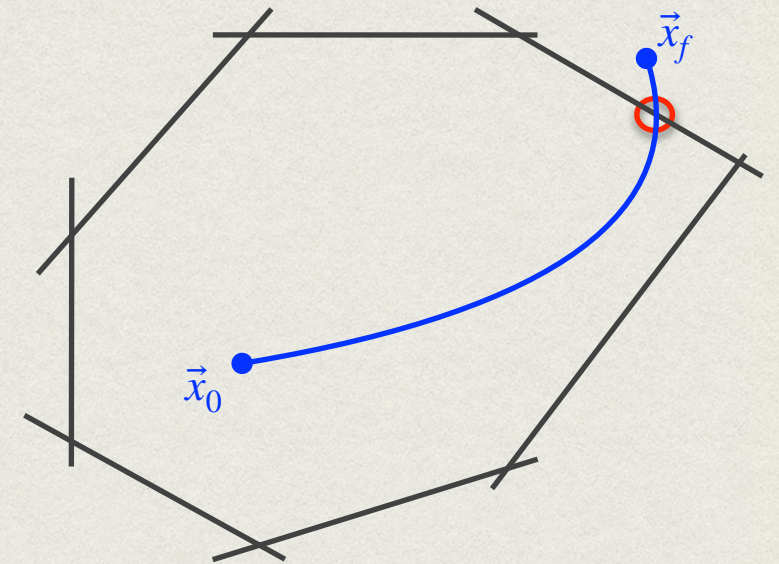
Evaluation of Feynman integrals

- Evaluation of Feynman integrals by the method of differential equations

$$\partial_x \vec{I}(\vec{x}; \epsilon) = A_x(\vec{x}; \epsilon) \vec{I}(\vec{x}; \epsilon)$$

- Find DEQs of the form

$$d A^{(\mathbf{X})}(\vec{x}; \epsilon) = \sum_{k=0}^2 \epsilon^k \left[\sum_{\alpha} c_{k\alpha}^{(\mathbf{X})} d\log(W_{\alpha}(\vec{x})) + \sum_{\beta} d_{k\beta}^{(\mathbf{X})} \omega_{\beta}(\vec{x}) \right]$$



$$\omega_{\beta}(\vec{x}) = \omega_{\beta}^{(s)}(\vec{x})ds + \omega_{\beta}^{(t)}(\vec{x})dt + \omega_{\beta}^{(m^2)}(\vec{x})dm^2 + \omega_{\beta}^{(q^2)}(\vec{x})dq^2$$

Move **difficult** integrals to very late stages

Available Mathematica implementations (DiffExp, ...) but slow for MC evaluations and costly to generate grids

- Our numerical approach $\mathcal{O}(10 - 100\text{ms})$ per point

[Petit Rosas, WJT (2025)]

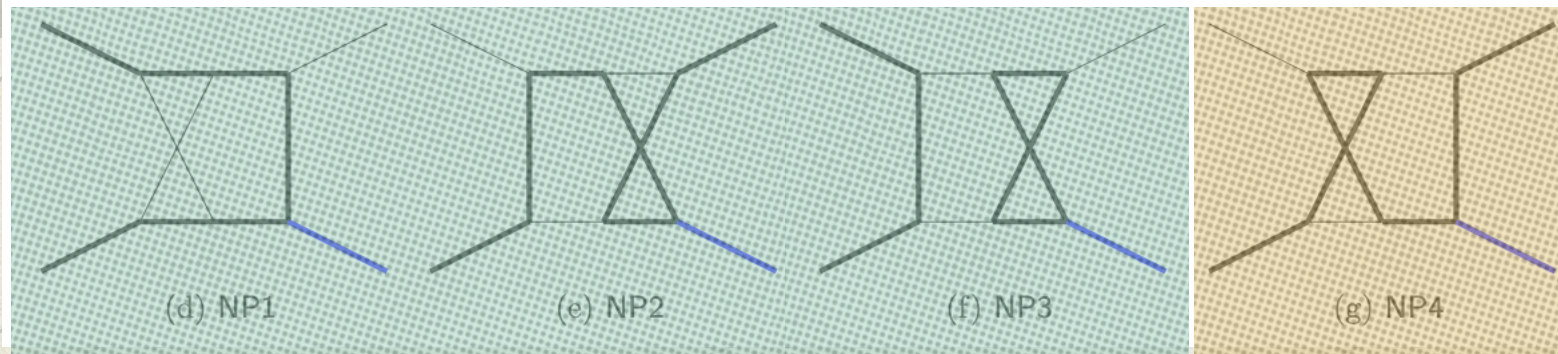
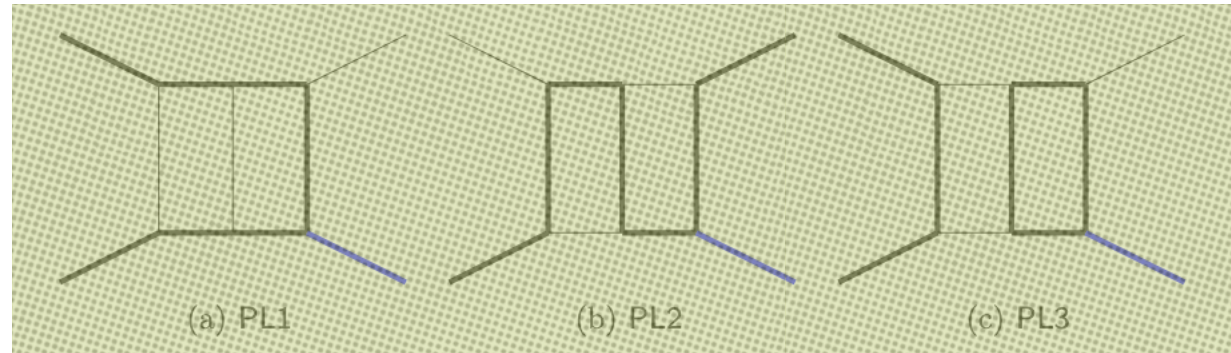
- ✓ Fast integrator in a low-level language
- ✓ Precise, but no need for 50 significant figures!
- ✓ Ideally, fast enough to not use grids

Validations

- ✓ $e^+e^- \rightarrow \pi^+\pi^-\gamma$ up $\mathcal{O}(\epsilon^2)$
- ✓ $pp \rightarrow ttj$ @ 2L
- ✓ $e^+e^- \rightarrow \gamma\gamma^*$ @ 2L

Evaluation of Feynman integrals for $e^+e^- \rightarrow \gamma\gamma^*$ @ $2L$

[In collaboration with A. Matijasic, D. Melnichenko, M. Pozzoli and S. Weinzierl]



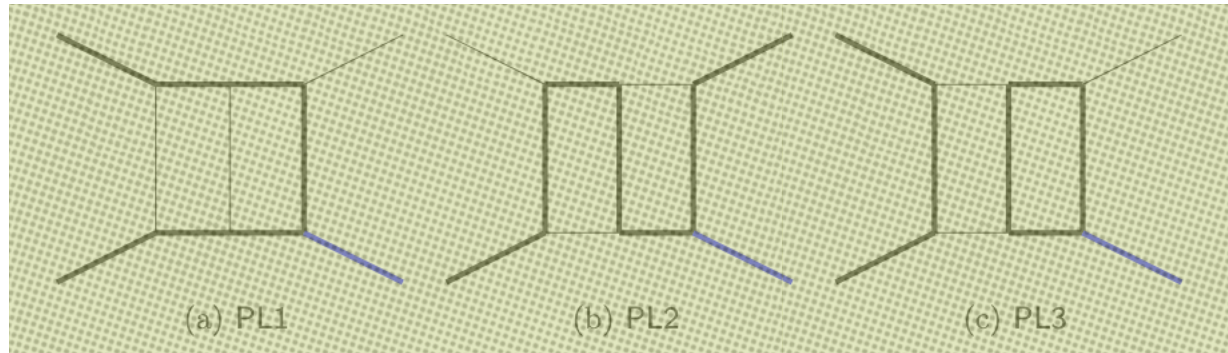
Completed [Pozzoli, WJT (2026)]

Found MIs

Work in progress

Evaluation of Feynman integrals for $e^+e^- \rightarrow \gamma\gamma^*$ @ 2L

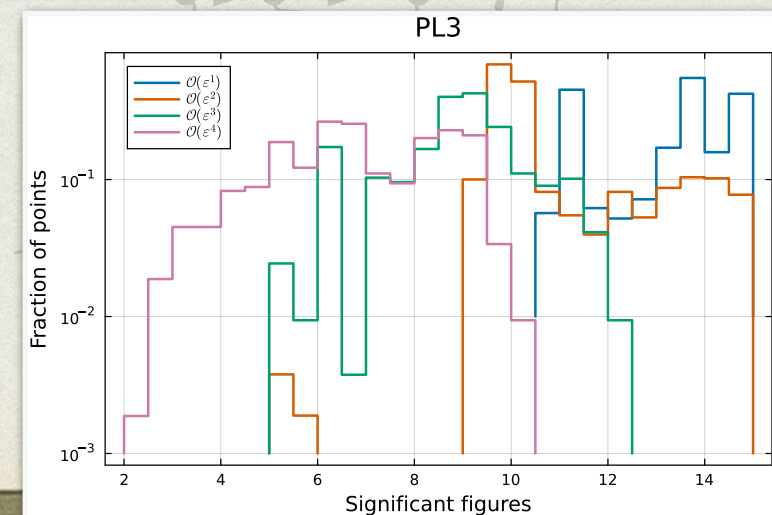
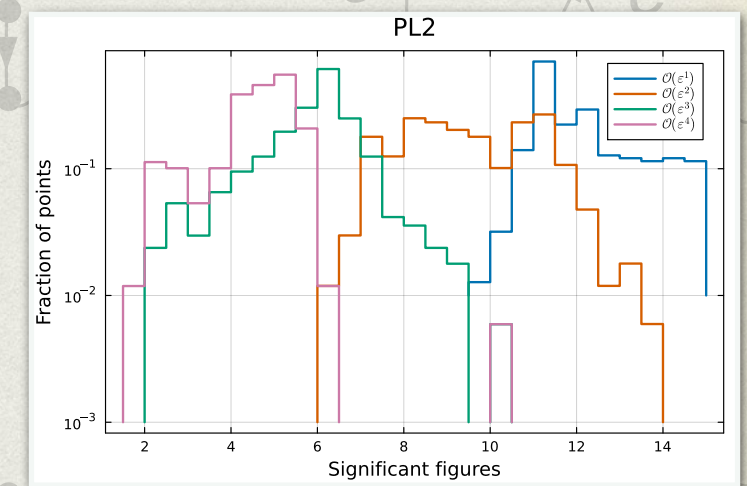
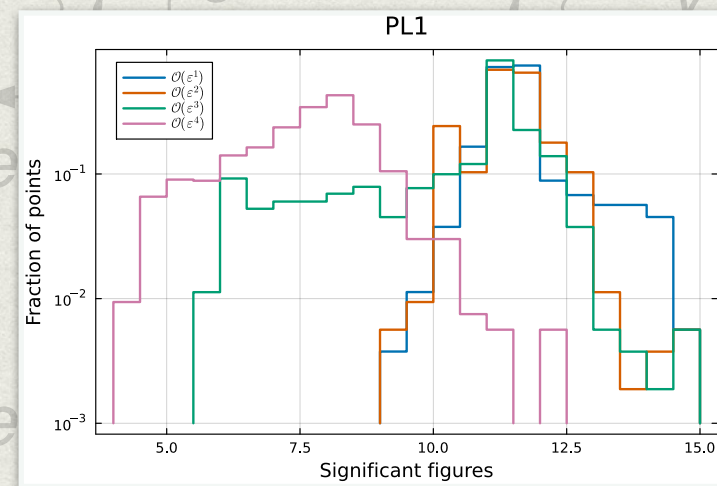
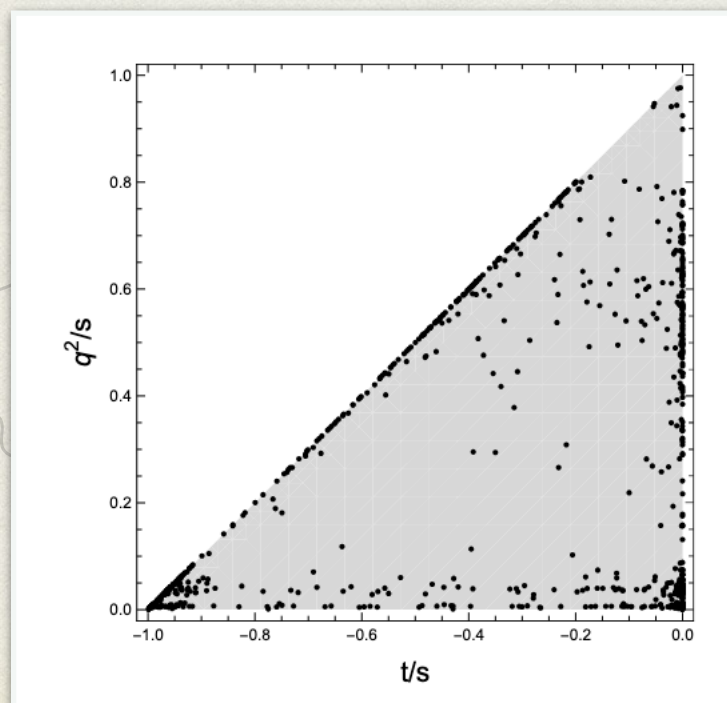
[In collaboration with A. Matijasic, D. Melnichenko, M. Pozzoli and S. Weinzierl]



Completed [Pozzoli, WJT (2026)]

✓ Numerical evaluation depending on the phase-space point $\mathcal{O}(40 - 800 \text{ ms})$

Distribution of 1000 phase-space points generated with PHOKHARA for $s \in \{1.0404, 16, 100\} \text{ GeV}$



We're home and dry — let's conclude!

Conclusions

📌 We have reached (and will be discussed in more details):

- ☑ Analytic study of radiative return processes
- ☑ First improvements and validation of the Phokhara generator
GVMD + Soft-photon resummation
- ☑ First look at the evaluation of two-loop Feynman integrals for $e^+e^- \rightarrow \gamma \gamma^*$

📌 Short term plans

- ☐ Complete radiative return NLO resummed for KLOE scenarios
- ☐ Complete calculation of two-loop Feynman integrals for $e^+e^- \rightarrow \gamma \gamma^*$

📌 Outlook

- ☐ Implement ISR NNLO in PHOKHARA
- ☐ Explore additional models for Pion's internal dynamics in PHOKHARA

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